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# Nonlinear Waves

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## Abstract

A method for solving a quasilinear nonelliptical equation of the second order is developed and we give classification and parametrization of simple elements of the equation. The equation for potential stationary flow of a compressible gas in a supersonic region is considered as the first example. A new exact solution is obtained which may be treated as a nonlinear analogue of a stationary wave. A gauge structure for the equation and an analogue of the Bäcklund transformation are introduced. Certain classes of the exact solution for equation of nonstationary potential flow of a compressible gas are found. This is the second example. Finally a physical analysis of the results is carried out.

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## 1. Introduction

In this work we present a method for solving nonlinear and nonelliptical partial differential equations. The paper is devoted to the theory of a differential equation of second order with coefficients depending on first derivatives. The method is old, for we employ Riemann invariants which have in fact been known for a long time [1, 2, 3]. Nevertheless, we have found a new feature of this old theory related to the problem of the algebraization of a differential equation.

We calculate exactly a characteristic cone for a Riemann wave and classify all simple elements for such an equation. All these simple elements depend on parameters and we proved that these parameters are parameters of some orthogonal or pseudo-orthogonal groups (which is very important in a later analysis (see section 6)).

Having classified the simple elements we may classify simple waves. Every simple element belongs to some submanifold  $F_i \subset \mathcal{E}^*$ . The dimension of this submanifold equals the number of parameters and these parameters are arbitrary functions of dependent variables. Since we are dealing with simple waves, the parameters become functions of  $R$  (Riemann invariant). Due to these arbitrary functions we may integrate the equations for simple waves, and this indicates that it is possible to introduce a gauge structure for every class of simple elements.

The paper is divided into 10 sections. In the second section we describe a parametrization and a classification of simple elements and we write down all possible elements. In the third section we deal with the first example from gas-dynamics. It is the equation of potential stationary flow for a perfect gas in a supersonic region. In the fourth section we calculate all possible simple elements for this equation and we collect them in the Appendix A (Table 1, 2, 3). In section 5 we calculate the simple wave corresponding to one of the simple elements, find a new exact solution of the equation and examine its properties. The solution depends on three arbitrary functions. We shift the arbitrariness from the parameters

(depending on  $R$  – Riemann invariant) to more convenient functions. This results in some restrictions on the arbitrary functions and on the range of the Riemann invariant  $R$ . The new freedom associated with the parameters of the characteristic cone, and the restriction imposed on them are new points in the Riemann invariants method. Simultaneously we obtain a restriction on the range of the dependent variable,  $R$ . This seems to be a new feature also. We derive the solution for the equation of potential stationary flow of the perfect gas (in three dimensions). This solution depends on three arbitrary functions and restrictions which we have to impose on the functions lead to physical effects. The solution may be considered as an analogue of a nonlinear stationary wave and we obtain planes of “density and pressure nodes” and planes of “magnitude of velocity antinodes”. In section 6 we deal with a “gauge” structure for the equation and its simple waves and we derive a transformation of gauge type connecting two simple waves (from the same class). The transformation may be treated as a nonlinear representation of a gauge group originating from the orthogonal or pseudo-orthogonal group. The transformation is similar to the classical Bäcklund transformation. In section 7 we deal with the second example from gas dynamics i.e. with an equation for a nonstationary flow of compressible gas described by a potential of velocity and density. In the eight section we find simple integral elements for this equation, which are presented in Appendix B. Section 9 is devoted to simple waves for this equation with some physical interpretation. We find several classes of exact solutions.

## 2. The algebraization procedure

In this section we consider the algebraization procedure.

Let us consider a system of partial differential equations

$$(2.1) \quad \begin{aligned} & \nu=1, 2, \dots, n. \\ & a_j^{s\nu}(u^1, u^2, \dots, u^l) \frac{\partial}{\partial x^\nu} u^j(x^1, x^2, \dots, x^n) = 0 \quad s=1, 2, \dots, m. \quad m \geq l \\ & j=1, 2, \dots, l. \end{aligned}$$

$$x = (x^1, x^2, \dots, x^n) \in \mathcal{E}, \quad u(x) = (u^1(x), u^2(x), \dots, u^l(x)) \in \mathcal{H}$$

which is a quasilinear homogeneous system of first order with coefficients depending only on the unknown functions. We suppose that  $a_j^{s\nu}(u)$  are smooth functions of  $u$  (at least of  $C^1$  class) in a certain open set  $\vartheta \subset \mathcal{H}$ . This system may be over-determined i.e.  $m \geq l$ . Let us suppose that this system is a nonelliptical one. This means that there exist some nontrivial solutions of the “algebraic system” of equations:

$$(2.2) \quad a_j^{s\nu} \gamma^j(u_0) \lambda_\nu(u_0) = 0 \quad \text{where} \quad \text{rank} \|a_j^{s\nu}(u_0) \lambda_\nu(u_0)\| < l, \quad u_0 \in \mathcal{H} = R^l$$

for vectors  $\gamma \in \mathcal{H} = R^l$  and  $\lambda \in \mathcal{E}^* = R^n$ .

The above algebraic system of equations defines respectively so-called knotted characteristic vectors in a *hodograph* space  $\mathcal{H} = R^l$  (i.e., the space of the values of the functions  $u^j$ ) and in a physical space  $\mathcal{E} = R^n$  (i.e., the space of independent variables). The pair  $\gamma$  and  $\lambda$  will be called a knotted pair for  $u_0 \in \mathcal{H}$  iff it obeys the equation (2.2). This fact will be denoted by  $\gamma(u_0) \sim \lambda(u_0)$ . The matrix  $L_\nu^j(u_0) = \gamma^j(u_0) \lambda_\nu(u_0)$  created by a pair of knotted vectors will be a simple integral element, because  $\text{rank} \|L_\nu^j(u_0)\| = 1$ , where  $u_0 \in \mathcal{H}$ .

In this case we can consider several notions of hyperbolicity. We say that the system (2.1) is hyperbolic at a point  $u_0 \in \mathcal{H}$  in a direction  $\mathcal{E}^* \ni v \neq 0$  if for every  $\mathcal{E}^* \ni w \neq 0$  the equation

$$\det \|a_j^{s\nu}(u_0)(v_\nu + \zeta w_\nu)\| = 0 \quad (\text{iff } l = m)$$

has only real solutions for  $\zeta$  or more generally if

$$\text{rank} \|a_j^{s\nu}(u_0)(v_\nu + \zeta w_\nu)\| < l$$

has only real solutions. The system is hyperbolic at a point  $u_0$  if the system is hyperbolic at  $u_0$  for any nonzero  $v \in \mathcal{E}^*$ . The last means that (2.2) has real and only real solutions. The system is hyperbolic in an open set  $\vartheta \subset \mathcal{H} = R^l$  if it is hyperbolic at any point of the set  $\vartheta$ . This means also that the system is hyperbolic in a usual sens i.e. it has only real characteristics.

Strong hyperbolicity means that any integral element  $L_\nu^j(u_0)$  annihilated by the matrix  $a_j^{s\nu}$ , i.e.  $a_j^{s\nu}(u_0)L_\nu^j(u_0) = 0$  must be of the form

$$L_\nu^j(u_0) = \sum \xi^s \gamma^j(u_0) \lambda_\nu(u_0)$$

and a system is hyperbolic in an open set  $\vartheta \subset \mathcal{H} = R^l$ . We call the system of equations (2.1) hyperbolic in a strong sense if all its integral elements are generated by simple elements

$$du = \sum_{s=1}^n \xi^s(x) \gamma_s(u) \otimes \lambda^s(u) = L_\nu^j dx^\nu.$$

If the matrix in (2.2) is square, i.e.  $m = l$  we get a solvability condition in a simpler form, i.e.

$$\det \|a_j^{s\nu} \lambda_\nu\| = 0.$$

It is convenient to consider  $\lambda$  as an element of the space  $\mathcal{E}^*$  which is the space of linear forms,  $\mathcal{E}^* \ni \lambda : \mathcal{E} \rightarrow R^1$ . On the other hand, in the terminology of tensor calculus if we consider  $x \in \mathcal{E}$  as a contravariant vector then  $\lambda \in \mathcal{E}^*$  is a covariant vector. In these terms the element  $L$  is an element of the tensor space  $T_u \mathcal{H} \otimes \mathcal{E}^*$  of the form  $L = \gamma \otimes \lambda$ . Now we introduce a simple wave, which suggests a separation of simple integral elements from a set of integral elements. In order to do this we formulate the theorem (which allows us to define a simple wave).

**THEOREM–DEFINITION.** *Let the mapping  $u : D \rightarrow \mathcal{H}$ ,  $D \subset \mathcal{E}$  be any solution of the system (2.1). We call  $u$  a simple wave for a homogeneous system if the tangent mapping  $du$  is a simple element at any point  $x_0 \in D$ . Let us consider the smooth curve  $\Gamma : R \rightarrow f(R)$  in the hodograph space  $\mathcal{H}$  parametrized by  $R$ , so the tangent vector such that*

$$(2.3) \quad \frac{df(R)}{dR} = \gamma(f(R))$$

is a characteristic vector. Then there exists a field of characteristic covectors  $\lambda(u)$  connected with  $\gamma(f(R))$  specified on the curve  $\Gamma : \lambda = \lambda(f(R))$ . Hence we may state the following. If the curve  $\Gamma \subset \mathcal{H}$  obeys the condition (2.3) and if  $\varphi(\cdot)$  is any differentiable function of one variable, then the function  $u = u(x)$  specified in an implicit way by relations (equations)

$$(2.4) \quad \begin{cases} u = f(R) \\ R = \varphi(\lambda_\nu(f(R))x^\nu) \end{cases} \quad \text{where } a_j^{s\nu} \gamma^j \lambda_\nu = 0$$

is the solution of the basic system (2.1). The solution is called a simple wave or Riemann wave. (The second equation of (2.4) is not always solvable so one should impose certain conditions on  $\varphi$  and  $f$ . Moreover it is always solvable locally i.e. in a certain open set  $\mathcal{O} = (R_0, R_1) \times \mathcal{O}'$ ,  $R_0, R_1 \in R^1$ ,  $\mathcal{O}' \subset \mathcal{E}$  under given assumptions by an implicit function theorem in  $R^{n+1}$ .)

A proof may be obtained by direct differentiation of the implicit relations (2.4). The covector  $\lambda$  in (2.4) specifies the velocity and direction of the wave propagation. The curve  $\Gamma$  fulfilling the condition (2.3) is called a characteristic curve in the hodograph space,  $\mathcal{H}$ . If the mapping  $u : \mathcal{E} \rightarrow \mathcal{H}$  is a simple wave, then a characteristic curve in space  $\mathcal{H}$  is the image of the map  $u$ . The parameter  $R$ , specified on this curve, is called a Riemann invariant.

It is interesting to mention that on the hypersurface  $S$  in the space  $\mathcal{E}$  given by two equations:

$$\begin{aligned} R &= \varphi(\lambda_\mu(f(R))x^\mu), \\ 1 - \dot{\varphi} \frac{d\lambda_\mu(f(R))}{dR} x^\mu &= 0, \end{aligned}$$

the gradient of the function  $R(x)$  becomes infinite (that is the so-called gradient catastrophe), i.e. the solution does not make sense on the hypersurface  $S$ . In this case some discontinuities can take place on the hypersurface  $S$  — e.g. shock waves. The covector  $\lambda$  is orthogonal to a level surface of the function  $R(x)$ . We can define a one-parameter family of hyperplanes. by an implicit equation  $r = \varphi(\lambda_\mu(r)x^\mu)$ , where  $r$  is a parameter of a family. In this way

$$r = \varphi(\lambda_\mu(r)x^\mu), \quad \text{where} \quad \lambda_\mu(r) = \lambda_\mu(f(r))$$



$$1 - \dot{\varphi} \lambda_{\mu,r}(r) \cdot x^\mu = 0, \quad \text{where} \quad \lambda_{\mu,r}(r) = \frac{d\lambda_\mu(f(r))}{dr}$$

are equations of an envelope of this family. So the hypersurface  $S$  is an envelope of the family  $R(x) = \text{const}$ . This hypersurface is the place for a gradient catastrophe.

Now let us consider a nonelliptical equation of the second order

$$(2.5) \quad \sum_{i,j=1}^n a_{ij} \left( \frac{\partial \Phi}{\partial x^1}, \frac{\partial \Phi}{\partial x^2}, \dots, \frac{\partial \Phi}{\partial x^n} \right) \frac{\partial^2 \Phi}{\partial x^i \partial x^j} = 0,$$

where  $\Phi : D \subset R^n \rightarrow R'$  is a real function of  $n$  variables of  $C^{(2)}$  class, and  $D$  a domain of  $\Phi$  is an open set. We suppose that  $a_{ij}$  are smooth (at least of  $C^1$  class) functions of their variables. Equation (2.5) may be transformed, by the introduction of new unknown functions, to the system of equations of the first order:

$$(2.6) \quad \begin{cases} \sum_{i,j=1}^n a_{ij}(u^1, \dots, u^n) \frac{\partial u^i}{\partial x^j} = 0 \\ \frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i} = 0 \end{cases}$$

where

$$u^i = \frac{\partial \Phi}{\partial x^i} \quad i = 1, 2, \dots, n.$$

Rewriting the system of equations (2.6) in terms of simple integral elements we have:

$$(2.7a) \quad \sum_{i,j=1}^n a_{ij} \gamma^i \lambda_j = 0$$

$$(2.7b) \quad \gamma^i \lambda_j - \gamma^j \lambda_i = 0 \quad i, j = 1, 2, \dots, n$$

From equation (2.7b) we find that the vector  $\lambda$  is proportional to vector  $\gamma$ . Thus from equation (2.7a) we get a quadratic form with respect to the variables  $\lambda_1, \lambda_2, \dots, \lambda_n$ :

$$(2.8) \quad Q(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i,j=1}^n a_{ij} \lambda_i \lambda_j = 0.$$

Equation (2.8) is the equation of a cone of the characteristic covector  $\lambda$ , specifying the velocity and direction of propagation of the simple wave.

Our aim will be to find the parametrical equations of this cone and at the same time to parametrize the covector  $\lambda$ .

To do this we transform the quadratic form  $Q$  to a canonical form, i.e., we diagonalize the symmetrized matrix  $A = (\tilde{a}_{ij})$  of the form  $Q$ . Hence we look for eigenvalues of this matrix and write the secular equation

$$(2.9) \quad \det (A - \omega I) = 0$$

We also search for a matrix  $B$  which diagonalizes the matrix  $A$

$$(2.10) \quad B^T A B = \begin{pmatrix} \omega_1 & & & 0 \\ & \omega_2 & & \\ & & \ddots & \\ 0 & & & \omega_n \end{pmatrix}, \quad B^T = B^{-1}$$

where  $\omega_1, \omega_2, \dots, \omega_n$  are the eigenvalues of the matrix  $A$ . The matrix  $A$  is a symmetric real matrix and it has real eigenvalues. There always exists at least one such orthogonal matrix  $B$  responsible for a rotation in  $n$ -dimensional Euclidean space from variables  $V = (V^1, \dots, V^n)$  to variables  $\lambda = (\lambda^1, \dots, \lambda^n)$

$$(2.11) \quad \lambda = B \begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ V^n \end{pmatrix} = B V$$

and

$$Q(\lambda_1, \dots, \lambda_n) = \omega_1 (V_1)^2 + \omega_2 (V_2)^2 + \dots + \omega_n (V_n)^2 = (B^T A B)_{ij} V_i V_j$$

We assume that  $\omega_i \neq 0$ ,  $i = 1, 2, \dots, n$ . The case with  $\omega_i = 0$  for some  $i$  is examined separately.



The same may be said about the remaining subsets of variables  $V_i$ , i.e. sequences of  $l_i$  elements.

The action of the  $l_i \times l_i$  matrix  $C_i^T = C_i^{-1}$  (the orthogonal one) cannot destroy the diagonalization. It is easy to see that the matrix  $C_i$  corresponds to the following  $n \times n$  matrix  $B_i$ ,

$$(2.14) \quad B_i = \underbrace{\left( \begin{array}{ccccccc} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & q_1 & & 1 & & & \\ & & & \underbrace{\overbrace{\boxed{C_i}}^{l_i}}_{l_i} & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ 0 & & & & q_2 & & \ddots \\ & & & & & & 1 \end{array} \right)}_n \quad \Bigg) \Bigg\}^n$$

where

$$q_1 = \sum_{j=1}^{i-1} l_j, \quad q_2 = \sum_{j=i+1}^k l_j.$$

Hence the matrix  $B$  is defined *modulo* the product of  $K$  matrices  $B_i$  of the form (2.14). From that we obtain

$$(2.15) \quad B = B_0 \prod_{i=1}^k B_i$$

where  $B_0$  is an arbitrary, but established, matrix obeying (2.12). The order of multiplication of matrices  $B_i$  is irrelevant, because all matrices commute with each another. So we see that the problem of the parametrization of  $B$  is reduced to the parametrization of each of the  $K$  orthogonal matrices  $C_i$ . Each of these matrices is responsible for an arbitrary rotation or reflection in  $l_i$ -dimensional space, thus it depends on  $\frac{1}{2} l_i(l_i - 1)$  parameters which are generalized Euler angles and some set of discrete parameters. So we have  $C_i = C_i(\alpha_{j_i}^i)$ , where  $j_i = 1, 2, \dots, \frac{l_i(l_i-1)}{2}$ , and where  $i = 1, 2, \dots, K$ .

By enumerating them in the order of their occurrence, i.e. first parameters of matrix  $C_1$ , then  $C_2$  etc., we can write:

$$(2.16) \quad B = B(\alpha_1, \alpha_2, \dots, \alpha_m, K_1, K_2, \dots, K_l)$$

where

$$m = \frac{1}{2} \sum_{i=1}^K l_i(l_i - 1)$$

and  $K_1, K_2, \dots, K_l$  correspond to reflections.

Finally, we have

$$(2.17) \quad \gamma \sim \lambda = B(\alpha_1, \alpha_2, \dots, \alpha_m, K_1, K_2, \dots, K_l) V$$

After having matrix  $B$  parametrized we deal with the form  $Q$  in coordinates  $V$ , and consequently with the equation of the characteristic cone in these coordinates. So we have:

$$(2.18) \quad \sum_{i=1}^K \omega_i \left( \sum_{\mu=1}^{l_i} V_{p_i+\mu}^2 \right) = 0.$$

Compare (2.12).

We find a parametric equation of this cone. In order to do this we write (2.18) in the following form

$$(2.18') \quad \sum_{j=1}^n \zeta_j V_j^2 = 0 \quad \text{where } \zeta_j \text{ is one of } \omega_i.$$

The equation (2.18') is the equation of an  $(n-1)$ -dimensional quadric in projective coordinates in the canonical form [4].

Let us suppose that  $V_{j_0} \neq 0$  and divide both sides of (2.18') by  $V_{j_0}^2$  and introduce new coordinates

$$\sigma_j^{(1)} = \frac{V_j}{V_{j_0}}, \quad j = 1, 2, \dots, n, \quad j \neq j_0.$$

One gets

$$(2.19) \quad \sum_{\substack{j=1 \\ j \neq i}}^n \zeta_j \sigma_j^{(1)2} + \zeta_{j_0} = 0$$

This equation can be written in parametric form

$$(2.20) \quad \begin{aligned} {}^{(1)}\sigma_j &= {}^{(1)}\sigma_j \left( {}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2} \right) \\ j &= 1, 2, \dots, n, \quad j \neq j_0 \end{aligned}$$

${}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}$  are internal parameters of the quadric whereas the functions  ${}^{(1)}\sigma_j \left( {}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2} \right)$  are expressed by either trigonometric or hyperbolic functions with respect to the type of quadric (2.19) according to F.Klein's classification [4]. We parametrize the vector  $V$ :

$$(2.21) \quad V_j = {}^{(1)}\sigma_j \left( {}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2} \right) {}^{(1)}\tau_{n-1} \quad , \quad j \neq j_0$$

$$V_{j_0} = {}^{(1)}\tau_{n-1} \quad , \quad \tau_{n-1} \neq 0$$

where  ${}^{(1)}\tau_{n-1}$  is a new parameter.

In the case when  $V_{j_0} = 0$  we have

$$(2.22) \quad \sum_{\substack{j=1 \\ j \neq j_0}}^n \zeta_j V_j^2 = 0$$

so in this case the dimension of quadric (2.22) is smaller than that of (2.18').

Similarly we choose  $v_{j_1} \neq 0, j_1 \neq j_0, j_1 = 1, 2, \dots, n$  and introduce new variables

$$(2.23) \quad {}^{(2)}\sigma_j = \frac{V_j}{V_{j_1}} \quad , \quad j \neq j_1 \quad , \quad j \neq j_0.$$

We obtain the equation (2.22) in the form

$$(2.24) \quad \sum_{\substack{j=1 \\ j \neq j_0 \\ j \neq j_1}}^n \zeta_j {}^{(2)}\sigma_j^2 + \zeta_{j_1} = 0.$$

Now we write its parametric form

$${}^{(2)}\sigma_j = {}^{(2)}\sigma_j \left( {}^{(2)}\tau_1, {}^{(2)}\tau_2, \dots, {}^{(2)}\tau_{n-3} \right)$$

(one parameter less than above) and get:

$$\begin{aligned}
 V_j &= \binom{(2)}{\sigma}_j \binom{(2)}{\tau}_1, \binom{(2)}{\tau}_2, \dots, \binom{(2)}{\tau}_{n-3} \binom{(2)}{\tau}_{n-2}, \quad j \neq j_1, \quad j \neq j_0 \\
 (2.25) \quad V_{j_1} &= \binom{(2)}{\tau}_{n-2}, \quad \binom{(2)}{\tau}_{n-2} \neq 0 \\
 V_{j_0} &= 0
 \end{aligned}$$

and we proceed likewise i.e. we consider the case  $V_{j_0} = 0 = V_{j_1}$  and choose  $V_{j_2} \neq 0$  etc. After  $k$  steps of this procedure we get the following equations for  $V$ :

$$\begin{aligned}
 V_j &= \binom{(k)}{\sigma}_j \binom{(k)}{\tau}_1, \binom{(k)}{\tau}_2, \dots, \binom{(k)}{\tau}_{n-(k+1)} \binom{(k)}{\tau}_{n-k}, \quad \binom{(k)}{\tau}_{n-k} \neq 0 \\
 \bigwedge_{\nu=0,1,\dots,k-2} V_{j_\nu} &= 0 \\
 (2.26) \quad V_{j_{k-1}} &= \binom{(k)}{\tau}_{n-k} \\
 \bigwedge_{\nu=0,1,2,\dots,k-1} j &\neq j_\nu
 \end{aligned}$$

It is easy to see that we exhaust all possibilities if  $n = k_{\max} + 1$ .

Then we have

$$\begin{aligned}
 V_r &= \binom{(n-1)}{\sigma} \cdot \binom{(n-1)}{\tau}_1, \quad \binom{(n-1)}{\tau}_1 \neq 0 \\
 (2.27) \quad V_{j_{n-1}} &= \binom{(n-1)}{\tau}_1 \\
 \bigwedge_{\nu=0,1,\dots,n-2} r &\neq j_\nu
 \end{aligned}$$

and the rest of  $V_i$  equals to zero. The last case where  $V = 0$  is not interesting.

At this stage we can present a full parametrization of the covector  $\lambda$ , which divides the set of  $\lambda$ 's into  $(n - 1)$  connected components. Since we have

$$\lambda = B(\alpha_1, \alpha_2, \dots, \alpha_m, K_1, K_2, \dots, K_l) V,$$

where  $\alpha_1, \alpha_2, \dots, \alpha_m$  are generalized Euler angles, and  $K_1, K_2, \dots, K_l$  are equal to 1 or  $(-1)$

and correspond to reflections we also have

$${}^{(1)}\gamma \sim {}^{(1)}\lambda = BV = B \begin{pmatrix} {}^{(1)}\sigma_1({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \\ {}^{(1)}\sigma_2({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \\ \vdots \\ {}^{(1)}\sigma_{j_0-1}({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \\ 1 \\ {}^{(1)}\sigma_{j_0+1}({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \\ \vdots \\ {}^{(1)}\sigma_n({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \end{pmatrix} \tau_{n-1} \quad j_0$$

(2.28)

$$\sim B(\alpha_1, \dots, \alpha_m, K_1, \dots, K_l) \cdot \begin{pmatrix} {}^{(1)}\sigma_1({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \\ {}^{(1)}\sigma_2({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \\ \vdots \\ {}^{(1)}\sigma_{j_0-1}({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \\ 1 \\ {}^{(1)}\sigma_{j_0+1}({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \\ {}^{(1)}\sigma_n({}^{(1)}\tau_1, {}^{(1)}\tau_2, \dots, {}^{(1)}\tau_{n-2}) \end{pmatrix} \quad j_0$$

where  $j_0$  is the number of the row in which 1 appears.

It is easy to see that  ${}^{(1)}\lambda$  depends on  $(m + n - 2)$  arbitrary parameters and on  $l$  integers equal to  $(\pm 1)$ . For  ${}^{(2)}\gamma$  we have:



$$(2.29) \quad \begin{matrix} {}^{(2)}\gamma \\ {}^{(2)}\lambda \end{matrix} \sim B(\alpha_1, \dots, \alpha_m, K_1, \dots, K_l) \cdot \begin{pmatrix} {}^{(2)}\sigma_1({}^{(2)}\tau_1, {}^{(2)}\tau_2, \dots, {}^{(2)}\tau_{n-3}) \\ {}^{(2)}\sigma_2({}^{(2)}\tau_1, {}^{(2)}\tau_2, \dots, {}^{(2)}\tau_{n-3}) \\ \vdots \\ {}^{(2)}\sigma_{j_0-1}({}^{(2)}\tau_1, {}^{(2)}\tau_2, \dots, {}^{(2)}\tau_{n-3}) \\ 0 \\ {}^{(2)}\sigma_{j_0+1}({}^{(2)}\tau_1, {}^{(2)}\tau_2, \dots, {}^{(2)}\tau_{n-3}) \\ \vdots \\ {}^{(2)}\sigma_{j_1-1}({}^{(2)}\tau_1, {}^{(2)}\tau_2, \dots, {}^{(2)}\tau_{n-3}) \\ 1 \\ {}^{(2)}\sigma_{j_1+1}({}^{(2)}\tau_1, {}^{(2)}\tau_2, \dots, {}^{(2)}\tau_{n-3}) \\ \vdots \\ {}^{(2)}\sigma_n({}^{(2)}\tau_1, {}^{(2)}\tau_2, \dots, {}^{(2)}\tau_{n-3}) \end{pmatrix}$$

which depends on  $(m + n - 3)$  arbitrary parameters and  $l$  integers equal to  $\pm 1$ . Note that the 0 appears in the  $j_0^{\text{th}}$  row and that the 1 appears in the  $j_1^{\text{th}}$  row.

In this way we find that  ${}^{(n-k)}\gamma$  depends on  $(m + n - (k + 1))$  free parameters and  $l$  integers equal to  $(\pm 1)$ .

Then in the column on which  $B(\alpha_1, \dots, \alpha_m, K_1, \dots, K_l)$  acts, zeroes appear at the places of numbers  $j_0, j_1, \dots, j_{k-2}$  and an integer 1 at the place of number  $j_{k-1}$ . The last element

$$(2.30) \quad \begin{matrix} {}^{(n-1)}\gamma \\ {}^{(n-1)}\lambda \end{matrix} \sim B(\alpha_1, \alpha_2, \dots, \alpha_m, K_1, \dots, K_l) \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ {}^{(n-1)}\sigma \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} r \\ \\ \\ j_{n-2} \end{matrix}$$

(where  ${}^{(n-1)}\sigma \neq 0$ ) has only zeros in column  $V$  except the places corresponding to the number  $r$  and number  $j_{n-2}$  and it depends on  $m$  arbitrary parameters and  $l$  integers equal to  $(\pm 1)$ .

The division and parametrization of simple elements is not unique, it depends on the choice of sequence  $j_0, j_1, \dots, j_{n-2}$ . For any other choice it will in general be different.

Now, let us consider the case where matrix  $A$  has a zero eigenvalue. Let  $\omega_1 = 0$  and have an order  $l_1$ . In such a case we may proceed as in the  $(n - l_1)$ -dimensional case starting with a parametrization of the quadric (in projective coordinates)

$$(2.31) \quad \sum_{j=l_1+1}^n \zeta_j V_j^2 = 0$$

assuming that  $V_1, V_2, \dots, V_{l_1}$  are arbitrary, i.e.  $V_j = \mu_j$ ,  $j = 1, 2, \dots, l_1$ .

Thus we get a classification of simple elements of eq.(2.5). Let  $F_i \subset \mathcal{E}^*$  be a set of all simple elements belonging to one of these classes. If  $\lambda \in F_i$  then  $\lambda$  is given by one of the formulas (2.28), (2.29) etc. Thus in this way we get the theorem:

**THE CLASSIFICATION THEOREM.** *Consider a partial differential equation of a nonelliptical type of the second order with coefficients depending on first derivatives of  $\Phi$  (i.e. quasilinear) and being smooth functions (at least of  $C^1$  class) of their variables,*

$$\sum_{i,j=1}^n a_{ij} \left( \frac{\partial \Phi}{\partial x^1}, \frac{\partial \Phi}{\partial x^2}, \dots, \frac{\partial \Phi}{\partial x^n} \right) \frac{\partial^2 \Phi}{\partial x^i \partial x^j} = 0$$

where  $\Phi: D \subset R^n \rightarrow R^1$  is a real function of  $n$  variables of  $C^{(2)}$  class, and  $D$  a domain of  $\Phi$  is an open set. All solutions of the above equation of rank 1 can be described by simple elements obtained according to the procedure mentioned above. Rank 1 solution means that

$$\text{rank} \left| \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \right| = 1$$

These solutions are called simple waves (Riemann waves).

An outline of the proof has been given above. Thus we have found many types of simple elements depending on various number of free parameters. Each of them has its own type of simple wave. Hence, according to relations (2.4) we have

$$(2.32) \quad \begin{aligned} u &= {}^{(i)}f(R^i) \\ R^i &= \varphi(\lambda_\nu^i x^\nu), \end{aligned}$$

where

$$\begin{aligned} \frac{df}{dR^i} &= \lambda^{(i)}(f^i(R^i)) \\ \lambda^{(i)} &\in F_i, \end{aligned}$$

and where  $i$  enumerates the types of simple waves admissible by the (2.5). Because of the free parameters, which are functions of  $u$ , and consequently of  $R^i$  we may integrate (2.32) and get its exact solutions. It is easy to see that  $F_i$  is a smooth submanifold of  $\mathcal{E}^*$  whose dimension is equal to the number of parameters. Thus we get the classification theorem for simple waves of the equation (2.5). The solutions of (2.3) (simple waves) may have a gradient catastrophe.

### 3. Equation for potential stationary flow of a compressible perfect gas (the first example of an application of the method)

This section is devoted to the equation of potential stationary flow of a compressible perfect gas. We consider the equation of potential of velocity for a stationary flow of perfect gas

$$\begin{aligned} (3.1) \quad & (c^2 - \Phi_x^2)\Phi_{xx} + (c^2 - \Phi_y^2)\Phi_{yy} + (c^2 - \Phi_z^2)\Phi_{zz} + \\ & - 2(\Phi_x\Phi_y\Phi_{xy} + \Phi_x\Phi_z\Phi_{xz} + \Phi_y\Phi_z\Phi_{yz}) = 0 \end{aligned}$$

where

$$\begin{aligned} (3.2) \quad & c^2 = c_0^2 - \frac{\kappa - 1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) \quad \text{and} \quad \vec{u} = \vec{\nabla}\Phi, \\ & \Phi: D \subset R^3 \rightarrow R \quad \text{and} \quad \Phi \in C_D^{(2)}, \quad D \text{ open} \end{aligned}$$

( $\Phi$  is the velocity potential and  $c$  the velocity of sound).

The equation (3.1) is interesting for us in the supersonic region

$$(3.3) \quad 0 \leq c^2 \leq \Phi_x^2 + \Phi_y^2 + \Phi_z^2 .$$

In this region the equation is nonelliptical and we have [5]

$$(3.4) \quad c_0^2 \left( \frac{2}{\kappa + 1} \right) \leq \Phi_x^2 + \Phi_y^2 + \Phi_z^2 \leq c_0^2 \left( \frac{2}{\kappa - 1} \right).$$

Now we look for the solutions of (3.1) provided that (3.4) is satisfied. Using the fact that for a perfect gas  $c^2 = \frac{\kappa p}{\rho_0}$  (the so called adiabatic sound), from the adiabatic equation  $p = a \rho_0^\kappa$ ,  $a = \text{const.}$  we can calculate  $\rho_0$  and  $p$  from (3.2)

$$(3.5) \quad \rho_0 = \left[ \frac{1}{a\kappa} \left( c_0^2 - \frac{\kappa - 1}{2} (\Phi_x^2 + \Phi_y^2 + \Phi_z^2) \right) \right]^{\frac{1}{(\kappa - 1)}}$$

$$p = \left[ \frac{1}{a\kappa} \left( c_0^2 - \frac{\kappa - 1}{2} (\Phi_x^2 + \Phi_y^2 + \Phi_z^2) \right) \right]^{\frac{\kappa}{(\kappa - 1)}}$$

We are interested in solutions of the equation (3.1) that are given in terms of the Riemann invariants. Following the method we transform (3.1), by introducing some new variables, into a quasilinear system of equations of the first order:

$$(3.6) \quad \begin{aligned} & (c^2 - \varphi_1^2) \varphi_{1,x} + (c^2 - \varphi_2^2) \varphi_{2,y} + (c^2 - \varphi_3^2) \varphi_{3,z} + \\ & - 2(\varphi_1 \varphi_2 \varphi_{1,y} + \varphi_1 \varphi_3 \varphi_{1,z} + \varphi_2 \varphi_3 \varphi_{2,z}) = 0 \\ & \varphi_{1,y} - \varphi_{2,x} = 0 \\ & \varphi_{1,z} - \varphi_{3,x} = 0 \\ & \varphi_{2,z} - \varphi_{3,y} = 0 \end{aligned}$$

where

$$(3.7) \quad \Phi_x = \varphi_1, \quad \Phi_y = \varphi_2, \quad \Phi_z = \varphi_3.$$

Thus equation (3.1) is transformed into the over-determined system of 4 equations for 3 functions.

Recall that the equation (3.1) can be derived in the following way from a mass conservation law

$$(3.8) \quad \text{div}(\rho_0 \vec{v}) = \rho_0 \text{div} \vec{v} + \vec{v} \cdot \vec{\nabla} \rho_0 = 0.$$

By means of the Euler equation

$$(3.9) \quad (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} p}{\rho_0} = -\frac{c^2 \vec{\nabla} \rho_0}{\rho_0}$$

we obtain

$$(3.10) \quad c^2 \operatorname{div} \vec{v} - \vec{v}(\vec{v} \cdot \vec{\nabla}) \vec{v} = 0 .$$

By introducing a potential according to equation (3.2) and substituting it to (3.10) we find equation (3.1).

#### 4. Simple integral elements (for the first example of an application of the method)

In this section we find simple integral elements for the equation of potential stationary flow of a compressible perfect gas.

Now we write the characteristic equation of the system (3.6) in the form (2.8) introducing the covector  $\lambda$ :

$$(4.1) \quad Q(\lambda_1, \lambda_2, \lambda_3) = (c^2 - \varphi_1^2)\lambda_1^2 + (c^2 - \varphi_2^2)\lambda_2^2 + (c^2 - \varphi_3^2)\lambda_3^2 + \\ - 2(\varphi_1\varphi_2\lambda_1\lambda_2 + \varphi_1\varphi_3\lambda_1\lambda_3 + \varphi_2\varphi_3\lambda_2\lambda_3) = 0.$$

Following the procedure described in section 2 we parametrize the covector  $\lambda$ . To do this we are going to diagonalize the form (4.1) and write the secular equation

$$(4.2) \quad \det(A - \mu I) = 0, \quad \text{where} \quad A = \begin{pmatrix} (c^2 - \varphi_1^2) & -\varphi_1\varphi_2 & -\varphi_1\varphi_3 \\ -\varphi_1\varphi_2 & (c^2 - \varphi_2^2) & -\varphi_2\varphi_3 \\ -\varphi_1\varphi_3 & -\varphi_2\varphi_3 & (c^2 - \varphi_3^2) \end{pmatrix}.$$

From (4.2) we get the third order equation for a value  $\mu$

$$(4.3) \quad (c^2 - \mu)^2 (c^2 - (\varphi_1^2 + \varphi_2^2 + \varphi_3^2) - \mu) = 0$$

Hence one obtains two different eigenvalues

$$(4.4) \quad \mu_1 = c^2, \quad \mu_2 = c^2 - (\varphi_1^2 + \varphi_2^2 + \varphi_3^2)$$

from which the first has order 2. Thus the quadratic form  $Q$  reduced to canonical form is:

$$(4.5) \quad Q(\lambda, \lambda) = c^2(y_1^2 + y_2^2) + (c^2 - (\varphi_1^2 + \varphi_2^2 + \varphi_3^2))y_3^2 = 0.$$

Now we find parametric equations for (4.5). Supposing that  $y_1 \neq 0$ , we simply get

$$(4.6) \quad \frac{X^2}{\left(\frac{a}{c}\right)^2} - Y^2 = 1 \quad \text{where} \quad X = \frac{y_2}{y_1}, \quad Y = \frac{y_3}{y_1}$$

and where  $a^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 - c^2 > 0$  (supersonic flow).

Then we have

$$(4.7) \quad Y = \text{sh } \rho, \quad X = \frac{a}{c} \text{ch } \rho$$

i.e. the parametric equation of a hyperbola.

If  $y_1 = 0$  then one gets

$$(4.8) \quad c^2 y_2^2 - a^2 y_3^2 = 0$$

$$(4.9) \quad y_2 = \varepsilon \frac{a}{c} y_3, \quad \varepsilon = \pm 1.$$

In our case there is only one eigenvalue of order larger than 1, i.e. 2. Thus the diagonalizing matrix  $B$  will depend on only one parameter  $\alpha$ ,  $B = B(\alpha)$ . The matrix  $B$  may easily be built from eigen vectors of the matrix  $A$  and one obtains

$$(4.10) \quad B(\alpha, l) = \begin{pmatrix} \frac{-1}{\chi_2} \left( \varphi_2 \sin \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \cos \alpha \right) & \frac{1}{\chi_2} \left( (-1)^K \varphi_2 \cos \alpha + (-1)^{K+1} \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha \right) & \frac{\varphi_1}{\chi_1} \\ \frac{1}{\chi_2} \left( \varphi_1 \sin \alpha - \frac{\varphi_2 \varphi_3}{\chi_1} \cos \alpha \right) & \frac{(-1)^{K+1}}{\chi_2} \left( \varphi_1 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha \right) & \frac{\varphi_2}{\chi_1} \\ \frac{\chi_2}{\chi_1} \cos \alpha & \frac{(-1)^K \chi_2}{\chi_1} \sin \alpha & \frac{\varphi_3}{\chi_1} \end{pmatrix}$$

$K = 0, 1, \quad l = (-1)^K$

where  $\alpha$  is a parameter depending on  $\varphi_1, \varphi_2, \varphi_3$  and

$$\chi_1^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 \quad , \quad \chi_2^2 = \varphi_1^2 + \varphi_2^2.$$

The integer  $l$  equals  $(\pm 1)$  and is associated with a reflection in two dimensional space, whereas  $\alpha$  is associated with a rotation. According to section 1 we have

$$(4.11) \quad \gamma^{(1)} \sim \lambda^{(1)} = B(\alpha, l) \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix}$$

and

$$(4.12) \quad \gamma^{(2)} \sim \lambda^{(2)} = B(\alpha, l) \begin{pmatrix} 0 \\ 1 \\ \sigma \end{pmatrix} \quad , \quad \sigma = \frac{\varepsilon a}{c}$$

Then by inserting (4.10) into (4.11) and (4.12) we get explicitly the form of the simple elements. These elements are presented in Table 1 ( $F_1$  and  $F_2$  of Appendix A).

According to the results given in section 2  $\lambda^{(1)}$  depends on two parameters  $\alpha$  and  $\rho$ , whereas  $\lambda^{(2)}$  depends only upon  $\alpha$ .

Let us consider the form (4.5) once more, now assuming that  $y_3 \neq 0$  and let us introduce

$$(4.13) \quad X = \frac{y_2}{y_3} \quad , \quad Y = \frac{y_1}{y_3}.$$

Then we have

$$(4.14) \quad X^2 + Y^2 = \left(\frac{a}{c}\right)^2$$

so

$$(4.15) \quad X = \left(\frac{a}{c}\right) \sin \rho \quad , \quad Y = \left(\frac{a}{c}\right) \cos \rho$$

i.e. a parametric equation of a circle, where  $\rho$  is a function of  $\varphi_i$ ,  $i = 1, 2, 3$ . If  $y_3 = 0$  we obtain  $y_1 = y_2 = 0$ , thus a zero case.

We have

$$(4.16) \quad \gamma_1 \sim \lambda^{(1')} = B(\alpha, l) \begin{pmatrix} Y \\ X \\ 1 \end{pmatrix}.$$

By inserting (4.10) and (4.15) into (4.16) we obtain the explicit form of the simple elements. They are presented in Table 2 ( $F_{1'}$  of Appendix A).  $\lambda^{(1')}$  depends on two parameters:  $\alpha$  and  $\rho$ .

However, in one particular case a degeneration occurs and for  $K = 0$  we get only one parameter  $\beta = \rho + \alpha$ , whereas for  $K = 1$ ,  $\rho - \alpha = \omega$ . In this case  $\lambda^{(1')}$  depends, really, on only one parameter.

Let us consider (4.5) again, assuming that  $y_2 \neq 0$ . We introduce new coordinates

$$(4.17) \quad Y = \frac{y_3}{y_2} \quad , \quad X = \frac{y_1}{y_2}$$

and obtain from (4.5)

$$(4.18) \quad \frac{Y^2}{\left(\frac{a}{c}\right)^2} - X^2 = 1$$

and in a parametric form

$$(4.19) \quad \begin{aligned} Y &= \frac{a}{c} \operatorname{ch} \rho \\ X &= \operatorname{sh} \rho. \end{aligned}$$

Thus we have

$$(4.20) \quad \gamma_{1''} \sim \lambda^{(1'')} = B(\alpha, l) \begin{pmatrix} X \\ 1 \\ Y \end{pmatrix}.$$

If  $y_2 = 0$  we obtain from (4.5)

$$(4.21) \quad c^2 y_1 - a^2 y_3 = 0$$

$$(4.22) \quad y_1 = \varepsilon \frac{a}{c} y_3 \quad , \quad \varepsilon^2 = 1$$

and

$$(4.23) \quad \gamma_{2''} \sim \lambda^{(2'')} = B(\alpha, l) \begin{pmatrix} \sigma \\ 0 \\ 1 \end{pmatrix} \quad \text{where} \quad \sigma = \frac{\varepsilon a}{c}.$$

Inserting (4.10) into (4.20) and (4.23) we derive the explicit form of the simple elements  $\lambda^{(1'')}$  and  $\lambda^{(2'')}$  which are presented in Table 3 ( $F_{1''}$  and  $F_{2''}$  of Appendix A).

Thus we have found here several types of simple elements which are used to construct solutions, i.e. simple waves and their interactions, the so-called double and multiple waves.



THEOREM 1. *All simple elements of the equation for a potential stationary flow of a perfect gas in a supersonic region fall into the classes:  $F_1, F_2, F_3, F_4, F_{1'}, F_{2'}, F_{1''}, F_{2''}, F_{3''}$ .*

In Appendix A we present all classes of simple elements –  $F_1, F_2, F_3, F_4, F_{1'}, F_{2'}, F_{1''}, F_{2''}, F_{3''}$  for (3.1).

## 5. Simple waves (the first example of an application of the method)

This section is devoted to simple waves for the equation of potential stationary flow of a compressible perfect gas.

Now we present the simplest solutions of the system (3.1) namely those that have been constructed on the basis of homogeneous simple integral elements. The method of finding these solutions is presented in papers [1], [2], [3]. In this section we deal with a method of solving equation (2.3), using the results of section 2 about the parametrization of simple elements.

The crucial point of our method is the freedom of choice of parameters occurring in simple elements. According to the terminology presented in [1], [2] the elementary solution of the homogeneous system has been called a simple wave. Those solutions may be interpreted as waves since they are moving disturbances, the profile of which changes in a course of propagation (a sign of this is the implicit form of the relation (2.4) for the  $R(x)$  function). The form of the solution of (2.4) suggests that the covector  $\lambda$  may be regarded as equivalent to the wave vector  $(\omega, \vec{k})$  which specifies the velocity and direction of propagation of the wave. The specific profile of a simple wave is explicitly determined by its initial data but there is a certain amount of freedom of choice of one free function, which is a function of one variable. The above remarks concern all simple waves which have been found.

There also exists another freedom, associated with parameters in simple elements in our

method. This freedom is of another kind and has a different origin. Using it we may integrate equation (2.3) and obtain solutions with  $(q + 1)$  arbitrary function of one variable, where  $q$  is the number of independent parameters in the simple elements. For all functions we obtain a certain restriction and it seems to be an interesting feature of the method.

The simple wave obeys conditions (2.3) and (2.4). By substitution of the simple integral element (4.11) into equation (2.3), we obtain for  $K = 0$  (i.e.  $\lambda_{(K=0)}^{(1)} = F_1$  see Table 1 of Appendix A)

$$\begin{aligned}
 \frac{d\varphi_1}{dR} &= \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_2} \left[ -\left( \varphi_2 \sin \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \cos \alpha \right) + \left( \varphi_2 \cos \alpha - \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha \right) \text{sh } \rho \right] + \frac{c\varphi_1}{\chi_1} \text{ch } \rho \\
 (5.1) \quad \frac{d\varphi_2}{dR} &= \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_2} \left[ \left( \varphi_1 \sin \alpha - \frac{\varphi_2 \varphi_3}{\chi_1} \cos \alpha \right) - \left( \varphi_1 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha \right) \text{sh } \rho \right] + \frac{c\varphi_2}{\chi_1} \text{ch } \rho \\
 \frac{d\varphi_3}{dR} &= \frac{\chi_2(\chi_1 - c^2)^{1/2}}{\chi_1} \left[ \cos \alpha + \sin \alpha \text{sh } \rho \right] + \frac{c\varphi_3}{\chi_1} \text{ch } \rho.
 \end{aligned}$$

Now we introduce the new dependent variables  $\chi_1, \chi_2$  and  $\mu_0 = \frac{\varphi_1}{\varphi_2}$ . We get from the system (5.1):

$$\begin{aligned}
 \frac{d\chi_1}{dR} &= c \text{ch } \rho \geq c \\
 (5.2) \quad \frac{d\chi_2}{dR} &= \frac{1}{\chi_1} \left[ -\varepsilon_1(\chi_1^2 - \chi_2^2)^{1/2}(\chi_1^2 - c^2)^{1/2}(\cos \alpha + \sin \alpha \text{sh } \rho) + c\chi_2 \text{ch } \rho \right] \\
 \frac{d}{dR} \text{arc tg } \mu_0 &= \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_2} (\cos \alpha - \sin \alpha).
 \end{aligned}$$

Let us observe that the quantities  $\alpha$  and  $\rho$ , entering in the equations (5.2), are arbitrary functions of  $R$ . Hence, it is convenient to shift the arbitrariness from  $\alpha$  and  $\rho$  to  $\chi_1$  and  $\chi_2$ . The right-hand sides of the equations for  $\chi_1$  and  $\chi_2$  depend on  $\alpha$  and  $\rho$  but they do not contain derivatives of these functions. Thus assuming that  $\chi_1$  and  $\chi_2$  are arbitrary we

obtain equations for  $\alpha$  and  $\rho$ . These equations will express  $\alpha$  and  $\rho$  in terms of the functions  $\chi_1$  and  $\chi_2$  and their derivatives with respect to  $R$ . The condition of solvability of algebraic (or transcendental) equations for  $\alpha$  and  $\rho$  provides us with restrictions on  $\chi_1$  and  $\chi_2$ .

In this way the problem of solving algebraic equations for  $\chi_1$  and  $\chi_2$  is reduced to the problem of solving algebraic equations and searching for restrictions for the arbitrary functions  $\chi_1$  and  $\chi_2$ . We will derive a set of such restrictions. After expressing  $\alpha$  and  $\rho$  in terms of new arbitrary functions with restrictions we insert these relations into the third equation of (5.2) and in a covector (4.11). In this way, on the right-hand side of this equation we have the functions of  $R$  and we can integrate the equation

$$(5.3) \quad \mu_0 = \text{tg} \left( \int_{R_0}^R \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_2} (\cos \alpha - \sin \alpha) dR' + d_0 \right) \\ d_0 = \text{const.}$$

Solving equations

$$\mu_0 = \frac{\varphi_1}{\varphi_2} \quad , \quad \chi_1^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 \quad , \quad \chi_2^2 = \varphi_1^2 + \varphi_2^2$$

we find the solution which is a simple wave

$$(5.4) \quad \begin{cases} \varphi_1 = \varphi_1(R) \\ \varphi_2 = \varphi_2(R) \\ \varphi_3 = \varphi_3(R) \\ R = \Psi \left( \overset{(1)}{\lambda}_1 x + \overset{(1)}{\lambda}_2 y + \overset{(1)}{\lambda}_3 z \right). \end{cases}$$

It is easy to see that the conditions for  $\chi_1$  and  $\chi_2$  imply some restrictions on the range of parameter  $R$  and consequently (see (5.4)) a restriction on the function  $\Psi$ . This restriction is responsible for the fact that the function  $\Psi$  has its range in a certain subset of the real axis.

Now let us realize this programme. For the convenience of calculations we assume that

$$(5.5) \quad \chi_1^2 = e^{2H} \quad , \quad \chi_2^2 = e^{2G} \quad \text{where} \quad e^{2G} \leq e^{2H}$$

and we find restrictions for  $H$  and  $G$  ( $H$  and  $G$  are functions of  $R$ ).

The first restriction for  $e^{2H}$  is of course (3.4) and we have:

$$(5.6) \quad c_0^2 \left( \frac{2}{\kappa + 1} \right) \leq e^{2H} \leq c_0^2 \left( \frac{2}{\kappa - 1} \right).$$

By substituting (5.5) into (5.2)<sub>1</sub> and (5.2)<sub>2</sub> and then using the relations between trigonometric and hyperbolic functions we get:

$$(5.7) \quad \begin{aligned} \text{ch } \rho &= \frac{e^H}{c} \frac{dH}{dR} \\ t^2 - \frac{2AB}{B^2 + 1} t + \frac{A^2 - 1}{B^2 + 1} &= 0 \end{aligned}$$

where

$$(5.8) \quad \begin{aligned} A &= \frac{e^{(G+H)} \frac{d}{dR} (G - H)}{(e^{2H} - e^{2G})^{1/2} (e^{2H} - c^2)^{1/2}}, \\ B &= \text{sh } \rho = \varepsilon_2 \left( \frac{e^{2H}}{c^2} \left( \frac{dH}{dR} \right)^2 - 1 \right)^{1/2} \end{aligned}$$

$$t = \sin \alpha \quad , \quad \cos \alpha = \varepsilon (1 - t^2)^{1/2}$$

$$\varepsilon_2^2 = \varepsilon_3^2 = 1.$$

Thus, we get the first restrictions:

$$(5.9) \quad \begin{aligned} 1^0. \quad & 1 \leq \text{ch } \rho = \frac{e^H}{c} \frac{dH}{dR} \\ 2^0. \quad & |t| \leq 1. \end{aligned}$$

At the same time the quadratic equation (5.7)<sub>2</sub> must have real roots, so its discriminant cannot be negative. The latter condition yields

$$(5.10) \quad B^2 - A^2 + 1 \geq 0.$$

By inserting  $A$  and  $B$  into (5.10) we obtain the following condition

$$(5.11) \quad \begin{aligned} \alpha_1 z_1^2 - \alpha_2 z_2^2 + \alpha_3 z_1 z_2 &\geq 0 \\ z_1 &= \frac{dH}{dR} \quad , \quad z_2 = \frac{dG}{dR} \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= e^{4H} - e^{2(H+G)} - c^2 e^{2H} \\ \alpha_2 &= c^2 e^{2G} \geq 0 \quad , \quad \alpha_3 = 2c^2 e^{2G} \geq 0. \end{aligned}$$

Supposing that  $G \neq \text{const}$ , we introduce the new variable  $z_3 = \frac{z_1}{z_2}$  and we obtain:

$$(5.12) \quad \alpha_1 z_3^2 + \alpha_3 z_3 - \alpha_2 \geq 0.$$

Now we present the conditions for which (5.12) is always satisfied regardless of the value of  $z_3$ . By doing this we avoid differential inequalities that are hard to satisfy. These conditions are the following: the discriminant  $\Delta_1$  of the quadratic equation (5.12) must be non-positive, whereas the coefficient of  $z_3^2$  must be positive. So, we have:

$$(5.13) \quad 0 \geq \Delta_1 = -e^{4H} + c^2 e^{2G} + e^{2(G+H)} - c^2 e^{2H}$$

$$(5.14) \quad 0 < \alpha_1 = e^{4H} - e^{2(H+G)} - c^2 e^{2H}.$$

The condition (5.13) is stronger than (5.14) and it is sufficient to fulfill only (5.13). By inserting (5.5) and (3.2) into (5.13) we obtain

$$(5.15) \quad 0 < (\kappa + 1)e^{4H} + (\kappa - 3)e^{2(H+G)} - 2c_0^2(e^{2G} + e^{2H}).$$

Using (5.6) we easily conclude that  $2c_0^2 - (\kappa - 3)e^{2H} > 0$ , so finally we have:

$$(5.16) \quad e^{2G} \geq \left[ \frac{(\kappa - 1)e^{2H} - 2c_0^2}{2c_0^2 - (\kappa - 3)e^{2H}} \right] e^{2H}.$$

Let us turn now to condition 2<sup>0</sup>. We solve the quadratic equation with respect to  $t$  and we get

$$(5.17) \quad t_{1,2} = \frac{AB \pm (B^2 - A^2 + 1)^{1/2}}{B^2 + 1}.$$

We choose the root with the smaller modulus

$$(5.17') \quad t = \varepsilon_2 \frac{|A| \cdot |B| - (B^2 - A^2 + 1)^{1/2}}{B^2 + 1} \quad , \quad \varepsilon_2^2 = 1.$$

Now we prove:

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$$(5.18) \quad \frac{||A| \cdot |B| - (B^2 - A^2 + 1)^{1/2}|}{B^2 + 1} \leq 1$$

when  $B^2 - A^2 + 1 \geq 0$ .

P r o o f . Let us consider the two cases

$$(5.19) \quad \begin{array}{ll} \text{a)} & |A| \cdot |B| - (B^2 - A^2 + 1)^{1/2} \leq 0 \\ \text{b)} & |A| \cdot |B| - (B^2 - A^2 + 1)^{1/2} \geq 0. \end{array}$$

a) We start from an obvious inequality  $(|A| + |B|)^2 \geq 0$  and we have

$$(5.20) \quad (A^2 + B^2 + 2|A| \cdot |B|)(B^2 + 1) \geq 0$$

i.e.

$$(5.21) \quad A^2 B^2 + A^2 + B^4 + B^2 + 2|A| \cdot |B| + 2|A| \cdot |B| B^2 \geq 0$$

or

$$(5.22) \quad B^2 - A^2 + 1 \leq B^4 + 1 + A^2 B^2 + 2|A| \cdot |B| + 2B^2 + 2|A| \cdot |B|^3 = (B^2 + 1 + |A| + |B|)^2$$

and from this,  $\sqrt{B^2 - A^2 + 1} - |A| \cdot |B| \leq (B^2 + 1)$  or

$$(5.23) \quad \frac{|(B^2 - A^2 + 1)^{1/2} - |A| \cdot |B||}{B^2 + 1} \leq 1.$$

b) We have  $B^2 - A^2 + 1 \geq 0$  that is

$$(5.24) \quad A^2 \leq B^2 + 1 .$$

So

$$(5.25) \quad A^2 B^2 \leq B^2 (B^2 + 1) \leq (B^2 + 1)^2$$

$$(5.25') \quad |A \cdot B| \leq (B^2 + 1)$$

Since  $|AB| \geq (B^2 - A^2 + 1)^{1/2}$  we have

$$(5.26) \quad \left| \frac{|AB| - (B^2 - A^2 + 1)^{1/2}}{B^2 + 1} \right| \leq 1.$$

So the proof has been established.

Summing up, we see that  $|t| \leq 1$ , so  $2^0$  is satisfied.

Now let us turn to condition  $1^0$ . This is a differential inequality which may be solved by applying well known results [6]. We have

$$(5.27) \quad \frac{e^H}{c(e^H)} \frac{dH}{dR} \geq 1 \quad , \quad H(R_0) = H_0$$

and the function  $P$  obeys the equation

$$(5.27a) \quad \frac{1}{c(P)} \frac{dP}{dR} = 1 \quad \text{and} \quad P(R_0) = e^{H_0}.$$

Using the definition of  $c$  we can write (5.27) and (5.27a) in the following form

$$(5.28) \quad \frac{d}{dR} \left[ \sqrt{\frac{2}{\kappa-1}} \arcsin \left( \frac{\sqrt{\frac{\kappa-1}{2}} e^H}{c_0} \right) \right] \geq 1$$

and  $H(R_0) = H_0$

$$(5.28a) \quad \frac{d}{dR} \left[ \sqrt{\frac{2}{\kappa-1}} \arcsin \left( \frac{\sqrt{\frac{\kappa-1}{2}} P}{c_0} \right) \right] = 1$$

$$P(R_0) = e^{H_0}.$$

From (5.28a) we get

$$(5.29) \quad P(R) = c_0 \sqrt{\frac{2}{\kappa-1}} \sin \left[ \sqrt{\frac{2}{\kappa-1}} (R - R_0) + \arcsin \left( \sqrt{\frac{\kappa-1}{2}} \frac{e^{H_0}}{c_0} \right) \right]$$

We have

$$(5.30) \quad \sqrt{\frac{2}{\kappa-1}} \arcsin \left( \frac{\sqrt{\frac{\kappa-1}{2}} e^{H(R)}}{c_0} \right) \geq \sqrt{\frac{2}{\kappa-1}} \arcsin \left( \frac{\sqrt{\frac{\kappa-1}{2}} P(R)}{c_0} \right)$$

and

$$(5.31) \quad e^{H(R_0)} = P(R_0) = e^{H_0}$$

(see e.g. [6]) hence

$$(5.31a) \quad e^H \geq c_0 \sqrt{\frac{2}{\kappa-1}} \sin \left[ \sqrt{\frac{2}{\kappa-1}} (R - R_0) + \arcsin \left( \frac{\sqrt{\frac{\kappa-1}{2}} e^{H_0}}{c_0} \right) \right]$$

$$e^{H(R_0)} = e^{H_0}.$$

Simultaneously we have

$$(5.32) \quad c_0 \sqrt{\frac{2}{\kappa+1}} \leq e^{H(R)} \leq c_0 \sqrt{\frac{2}{\kappa-1}}$$

$$(5.33) \quad c_0 \sqrt{\frac{2}{\kappa+1}} \leq e^{H_0} \leq c_0 \sqrt{\frac{2}{\kappa-1}}$$

Since  $P(R_k) = c_0 \sqrt{\frac{2}{\kappa-1}}$ , for

$$(5.34) \quad R_k = R_0 + \sqrt{\frac{\kappa-1}{2}} \left( \frac{\pi}{2} + 2k\pi - \arcsin \left( \frac{1}{c_0} \sqrt{\frac{\kappa-1}{2}} e^{H_0} \right) \right)$$

$$k = 0, \pm 1, \pm 2, \dots$$

we have

$$(5.35) \quad e^{H(R_k)} = c_0 \sqrt{\frac{2}{\kappa-1}}.$$

We have of course

$$(5.36) \quad |R_{k+1} - R_k| = \sqrt{2(\kappa-1)} \pi$$

Hence, the function  $e^{H(R)}$  is defined in the interval  $[R_0, +\infty)$  for an established but arbitrary  $R_0$ .



At points  $R_k$ ,  $c^2 = 0$ , the magnitude of the vector of velocity of flow is maximal, and pressure and density simultaneously become equal to zero. Thus, the solution of the system (5.1) is

$$(5.37) \quad \begin{aligned} \varphi_1 &= \eta_2 e^G \sin K(R) \\ \varphi_2 &= \eta_2 e^G \cos K(R) \quad \eta_1^2 = \eta_2^2 = 1 \\ \varphi_3 &= \eta_1 \sqrt{e^{2H} - e^{2G}} \end{aligned}$$

where

$$K(R) = \int_{R_0}^R (\cos \alpha - \sin \alpha) \frac{(e^{2H} - c^2)^{1/2}}{e^G} dR' + c'$$

$$c' = \text{const} \quad , \quad R = \Psi \left( \overset{(1)}{\lambda}_1 x + \overset{(1)}{\lambda}_2 y + \overset{(1)}{\lambda}_3 z \right) \quad , \quad R \in [R_0, +\infty) = L.$$

The smooth function  $\Psi$  takes values only from the interval  $L$ . The conditions (5.16), (5.31a), (5.32) and (5.33) are implied on functions  $G$  and  $H$ .

For  $\cos \alpha$  and  $\sin \alpha$  we have the following expressions:

$$\sin \alpha = \varepsilon_1 \left\{ \frac{e^G \frac{d}{dR} (H - G) \left( \frac{e^{2H}}{c^2} \left( \frac{dH}{dR} \right)^2 - 1 \right)^{1/2} - \left[ \left( \frac{dH}{dR} \right)^2 (e^{2H} - e^{2G}) \frac{(e^{2H} - c^2)}{c^2} + \right.}{\frac{1}{c^2} e^H \left( \frac{dH}{dR} \right)^2 (e^{2H} - e^{2G})^{1/2} \cdot (e^{2H} - c^2)^{1/2}} \right.$$

$$\left. - \frac{e^{2G} \left( \frac{d}{dR} (H - G) \right)^2 \right]^{1/2}}{\left. \right\}}$$

$$(5.38) \quad \cos \alpha = \varepsilon_3 (1 - \sin \alpha)^{1/2}$$

$$(5.39) \quad \text{ch } \rho = \frac{e^H}{c} \frac{dH}{dR}$$

$$\text{sh } \rho = \varepsilon_2 \left( \frac{e^{2H}}{c^2} \left( \frac{dH}{dR} \right)^2 - 1 \right)^{1/2}$$

where  $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 1$ . Of course, expressions (5.38) and (5.39) should be inserted in the covector  $\overset{(1)}{\lambda}$ , as in expressions (5.37).

Now let us consider the following equation

$$(5.40) \quad R_k = \Psi(\overset{(1)}{\lambda}_1(R_k)x + \overset{(1)}{\lambda}_2(R_k)y + \overset{(1)}{\lambda}_3(R_k)z)$$

$$R_k \in L$$

where  $R_k$  is given by (5.34). For a given  $k$  equation (5.40) describes one of several planes with a normal vector  $\vec{\lambda}(R_k) = (\overset{(1)}{\lambda}_1(R_k), \overset{(1)}{\lambda}_2(R_k), \overset{(1)}{\lambda}_3(R_k))$ . If the equation  $R_k = \Psi(r)$  possesses  $n$  roots  $r_i$ ,  $i = 1, 2, \dots, n$ ,  $\Psi(r_i) = R_k$ , then there are planes:

$$(5.41) \quad \lambda_1(R_k)x + \lambda_2(R_k)y + \lambda_3(R_k)z = r_i$$

$$i = 1, 2, \dots, n.$$

Thus, they are parallel planes. When the function  $\Psi$  is single valued we have only one plane. For different  $k_1 \neq k_2$  the planes belonging to  $k_1$  do not have to be parallel and in general they cross each other. The sections of planes may also cross.

Note that (5.41) is a place at which both density and pressure disappear and the magnitude of the velocity reaches its maximal value. So in fact they are planes of nodes of density and pressure and of antinodes of magnitude of the velocity vector. Because planes may cross each other we also have straight lines and points of nodes and antinodes. Thus, the above solution may be treated as a nonlinear analogue of a standing wave.

Now let us turn to the case  $G = G_0 = \text{const.}$ ,  $\frac{dG}{dR} = 0$ . We must specify the condition for  $\Delta \geq 0$  (compare (5.11)). Then we have  $z_2 = 0$  and (5.11) is reduced to:

$$(5.42) \quad \alpha_1 z_1^2 \geq 0.$$

Supposing that  $z_1 \neq 0$ ,  $H \neq \text{const.}$  we obtain  $\alpha_1 \geq 0$ , which by using (5.14) is reduced to:

$$(5.43) \quad e^{2H} \geq \left( \frac{2}{\kappa - 1} \right) (c_0^2 + e^{2G_0})$$

and now we may repeat all the considerations concerning the conditions  $1^0$  and  $2^0$  and derive the same results as before with the substitution  $G = G_0 = \text{const}$ . The only difference will be another restriction for the lowest value of  $e^H$  and  $e^{H_0}$  and the absence of the restriction (5.16).

The case  $H = \text{const}$  leads to a nonphysical solution. From the equation  $e^H \frac{dH}{dR} = c \operatorname{ch} \rho \geq c$  we have that  $c = 0$ , which leads to a vanishing of density and pressure everywhere, thus to the absence of gas.

**THEOREM.** *The expressions (5.37–5.39) together with conditions (5.16), (5.31a), (5.32) and (5.33) describe an exact solution of equation (3.1). The solution has remarkable properties being a nonlinear analogue of a standing wave (see equations (5.40) and (5.41)).*

Proof has been given above.

The gradient catastrophe for the above solution takes place on a hypersurface  $S$  defined by two relations

$$R = \Psi(\lambda_1^{(1)}x + \lambda_2^{(1)}y + \lambda_3^{(1)}z)$$

and

$$1 = \dot{\Psi}(\lambda_1^{(1)}x + \lambda_2^{(1)}y + \lambda_3^{(1)}z) \cdot \left( \frac{d\lambda_1^{(1)}}{dR}x + \frac{d\lambda_2^{(1)}}{dR}y + \frac{d\lambda_3^{(1)}}{dR}z \right).$$

For all simple elements from Appendix A it is possible to repeat these considerations (see References [7, 8]).

Now let us consider simple waves generated by the simple element  $\lambda^{(2)}$  for  $K = 0$ , i.e.  $F_2$  (see Table 1 of Appendix A). In this case we have the following system of equations:

$$\begin{aligned} \frac{d\varphi_1}{dR} &= \frac{\varepsilon(\chi_1^2 - c^2)^{1/2}}{\chi_2} \left( \varphi_2 \cos \alpha - \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha \right) + \frac{c\varphi_1}{\chi_1} \\ (5.44) \quad \frac{d\varphi_2}{dR} &= \frac{-\varepsilon(\chi_1^2 - c^2)^{1/2}}{\chi_2} \left( \varphi_1 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha \right) + \frac{c\varphi_2}{\chi_1} \\ \frac{d\varphi_3}{dR} &= \frac{\varepsilon(\chi_1^2 - c^2)^{1/2}}{\chi_1} \cdot \chi_2 \sin \alpha + \frac{c\varphi_3}{\chi_1} \end{aligned}$$

where  $\chi_1^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2$ ,  $\chi_2^2 = \varphi_1^2 + \varphi_2^2$ ,  $\varepsilon^2 = 1$ .

By introducing new variables  $\chi_1$ ,  $\chi_2$ ,  $\nu_0 = \frac{\varphi_1}{\varphi_2}$  we can transform (5.44) to the following system:

$$(5.45a) \quad \frac{d\chi_1}{dR} = c = \left( c_0^2 - \frac{\kappa - 1}{2} \chi_1^2 \right)^{1/2}$$

$$(5.45b) \quad \frac{d\chi_2}{dR} = \frac{\varepsilon}{\chi_1} \{ (\chi_1^2 - c^2)^{1/2} (\chi_1^2 - \chi_2^2)^{1/2} \sin \alpha + c \chi_2 \}$$

$$(5.45c) \quad \frac{d}{dR} \arctan \nu_0 = \frac{-\varepsilon (\chi_1^2 - c^2)^{1/2}}{\chi_2} \cos \alpha.$$

We integrate the first equation of (5.45) and have:

$$(5.46) \quad \chi_1 = c_0 \sqrt{\frac{2}{\kappa - 1}} \sin \left( \sqrt{\frac{\kappa - 1}{2}} (R + c_1) \right)$$

where  $c_1 = \text{const.}$

In the remaining two equations of (5.45) there is an arbitrary function  $\alpha(R)$ . According to the accepted procedure we shift the arbitrariness from  $\alpha$  to  $\chi_2 = e^G$  and we find the restriction for the function  $G(R)$ . By introducing  $\chi_2 = e^G$  to the last two equations of (5.45) we can calculate from b)  $\sin \alpha$  and then  $\cos \alpha$ . After simple calculation we have the following formulae

$$(5.47) \quad \sin \alpha = \frac{\chi_1 e^G \frac{dG}{dR} - c e^G}{(\chi_1^2 - c^2)^{1/2} (\chi_1^2 - e^{2G})^{1/2}} =$$

$$\frac{e^G \left( \sqrt{\frac{2}{\kappa - 1}} \sin \left( \sqrt{\frac{\kappa - 1}{2}} (R + c_1) \right) \frac{dG}{dR} - \cos \left( \sqrt{\frac{\kappa - 1}{2}} (R + c_1) \right) \right)}{\left( \left( \frac{\kappa + 1}{\kappa - 1} \right) \sin^2 \left( \sqrt{\frac{\kappa - 1}{2}} (R + c_1) \right) - 1 \right)^{1/2} \left( c_0^2 \left( \frac{2}{\kappa - 1} \right) \sin^2 \left( \sqrt{\frac{\kappa - 1}{2}} (R + c_1) \right) - e^{2G} \right)^{1/2}}$$

$$\cos \alpha = \eta \frac{\left\{ c_0^2 \left( \frac{2}{\kappa-1} \right)^2 \sin^4 \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) - \left[ c_0^2 \cos^2 \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) + e^{2G} \left( 1 + \frac{dG}{dR} \right)^2 \right] \right\}}{\left( \left( \frac{\kappa+1}{\kappa-1} \right) \sin^2 \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) - 1 \right)^{1/2}}$$

$$(5.48) \quad \frac{\left\{ \left( \frac{2}{\kappa-1} \right) \sin^2 \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) - e^{2G} \frac{dG}{dR} \sqrt{\frac{2}{\kappa-1}} \sin \left( \sqrt{2(\kappa-1)} (R + c_1) \right) \right\}^{1/2}}{\left( \frac{2c_0^2}{\kappa-1} \sin^2 \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) - e^{2G} \right)^{1/2}}$$

For the function  $\chi_1$  we have the following restriction

$$(5.49) \quad c_0 \sqrt{\frac{2}{\kappa+1}} \leq \chi_1 \leq c_0 \sqrt{\frac{2}{\kappa-1}}.$$

By inserting (5.46) into (5.49) we obtain the following inequality

$$(5.50) \quad \sqrt{\frac{\kappa-1}{\kappa+1}} \leq \sin \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) \leq 1$$

and this introduces the following restriction for  $R$

$$(5.51) \quad \begin{aligned} 2k\pi + \arcsin \sqrt{\frac{\kappa-1}{\kappa+1}} &\leq \sqrt{\frac{\kappa-1}{2}} (R + c_1) \leq \frac{\pi}{2} + 2k\pi - \arcsin \sqrt{\frac{\kappa-1}{\kappa+1}} \\ k &= 0, \pm 1, \pm 2, \dots \end{aligned}$$

Simultaneously we achieve a certain restriction for the function  $G$  caused by the necessity that the modulus of the expression (5.47) is less than 1, i.e. a certain differential inequality

$$(5.52) \quad \frac{\left| e^G \left( \sqrt{\frac{2}{\kappa-1}} \sin \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) \frac{dG}{dR} - \cos \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) \right) \right|}{\left( \left( \frac{\kappa+1}{\kappa-1} \right) \sin^2 \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) - 1 \right)^{1/2} \left( \frac{2c_0^2}{\kappa-1} \sin^2 \left( \sqrt{\frac{\kappa-1}{2}} (R + c_1) \right) - e^{2G} \right)^{1/2}} \leq 1$$

By inserting the following notations into (5.52)

$$(5.53) \quad \sqrt{\frac{2}{\kappa-1}} c_0 e^G = e^{G_1} \text{ and } \beta = \sqrt{\frac{\kappa-1}{2}} (R + c_1)$$

we get

$$(5.54) \quad \left| \frac{\frac{d}{d\beta}(e^{-G_1} \sin \beta)}{\left(\left(\frac{\kappa+1}{\kappa-1}\right) \sin^2 \beta - 1\right)^{1/2} (\sin^2 \beta - e^{2G_1})^{1/2}} \right| \leq 1.$$

We are able to solve this inequality by the substitutions

$$(5.55) \quad X = \frac{\sin \beta}{e^{G_1}}, \quad \xi = \sin \beta.$$

We transform (5.54) to the following expression

$$(5.56) \quad \left| \frac{d}{d\xi}(\arctan(X + \sqrt{X^2 - 1})) \frac{2\xi(1 - \xi^2)^{1/2}}{\left(\frac{\kappa+1}{\kappa-1} \xi^2 - 1\right)^{1/2}} \right| \leq 1.$$

By substituting for convenience  $\eta = \xi^2$  and  $\frac{\kappa+1}{\kappa-1} = \rho_1$  we get:

$$(5.57) \quad \left| \frac{d}{d\eta}(\arctan(X + \sqrt{X^2 - 1})) \frac{\eta(1 - \eta)^{1/2}}{(\rho_1 \eta - 1)^{1/2}} \right| \leq \frac{1}{4}$$

and by making a final change of dependent variable

$$(5.58) \quad r = 2 \left[ (\sqrt{\rho_1} \arctan\left(\sqrt{\frac{\rho_1 - \eta}{\rho_1 - \rho_1 \eta}}\right) - \arctan\left(\sqrt{\frac{\rho_1 \eta - 1}{1 - \eta}}\right) \right]$$

i.e. such that

$$(5.59) \quad \frac{dr}{d\eta} = \frac{(\rho_1 \eta - 1)^{1/2}}{\eta(1 - \eta)^{1/2}}$$

we derive the following inequality

$$(5.60) \quad \left| \frac{d}{dr}(\arctan(X + \sqrt{X^2 - 1})) \right| \leq \frac{1}{4}$$

Let

$$(5.61) \quad (X(0) + \sqrt{X^2(0) - 1}) = f_0$$

and let

$$(5.62) \quad \frac{df_1}{dr} = -\frac{1}{4} \quad , \quad \frac{df_2}{dr} = \frac{1}{4} \quad , \quad f_1(0) = f_0 = f_2(0).$$

Then

$$(5.63) \quad f_1 = -\frac{1}{4} r + f_0 \quad , \quad f_2 = \frac{1}{4} r + f_0$$

and

$$(5.64) \quad -\frac{1}{4} r + f_0 \leq \arctan(X + \sqrt{X^2 - 1}) \leq \frac{1}{4} r + f_0$$

for  $r \geq 0$ .

By introducing  $\eta = \sin^2 \beta$  into (5.58) we get:

$$(5.65) \quad 0 = \sqrt{\rho_1} \arctan\left(\sqrt{\frac{\rho_1 \sin^2 \beta_0 - 1}{\rho_1 - \rho_1 \sin^2 \beta_0}}\right) - \arctan\left(\sqrt{\frac{\rho_1 \sin^2 \beta_0 - 1}{1 - \sin^2 \beta_0}}\right)$$

from which we get

$$(5.66) \quad \sin^2 \beta_0 = \frac{1}{\rho_1}$$

From (5.61) we obtain

$$(5.67) \quad f_0 = \arctan\left(\frac{1}{\rho_1 e^{2G_1(\beta_0)}} + \sqrt{\frac{1}{\rho_1 e^{2G_1(\beta_0)}} - 1}\right)$$

from which we have

$$(5.68) \quad 0 \leq e^{G_1(\beta_0)} \leq \frac{1}{\sqrt{\rho_1}}$$

and

$$(5.69) \quad \arctan\left(\frac{1}{\rho_1^2} + \sqrt{\frac{1}{\rho_1^2} - 1}\right) \leq f_0 \leq \frac{\pi}{2}.$$

Examining the function (5.58) for  $\frac{1}{\rho_1} \leq \eta \leq 1$  we get

$$(5.70) \quad 0 \leq r \leq 2 \left[ \sqrt{\rho_1} \frac{\pi}{2} - \frac{\pi}{2} \right] = \pi(\sqrt{\rho_1} - 1).$$

Now let us examine the expression  $-\frac{1}{4} r + f_0$ :

$$(5.71) \quad \frac{\pi}{2} \geq -\frac{1}{4} r + f_0 \geq -\frac{\pi}{4}(\sqrt{\rho_1} - 1) + \arctan\left(\frac{1}{\rho_1^2} + \sqrt{\frac{1}{\rho_1^2} - 1}\right) = g(\rho_1)$$

$$(5.72) \quad \rho_1 = \frac{\kappa + 1}{\kappa - 1}$$

and thus  $1 < \rho_1 < \infty$ ,  $\kappa > 1$ .

The function  $g(\rho_1)$  is decreasing  $\frac{dg}{d\rho_1} < 0$ , so

$$(5.73) \quad g(\rho_1) \leq \max(g(\rho_1)) = g(1) = \arctan 1 = \frac{\pi}{4}.$$

From this we have

$$(5.74) \quad \frac{\pi}{2} \geq -\frac{1}{4} r + f_0 \geq g(\rho_1) \geq \frac{\pi}{4}.$$

On the other hand we have, that

$$(5.75) \quad \frac{\pi}{4}(1 + \sqrt{\rho_1}) \geq \frac{1}{4} r + f_0 \geq \arctan\left(\frac{1}{\rho_1^2} + \sqrt{\frac{1}{\rho_1^2} - 1}\right) \geq \frac{\pi}{4}.$$

So we have deduced that in the inequality (5.64) the upper limit is constantly positive and is contained in the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . Also  $-\frac{1}{4} r + f_0$  is contained in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . Thus by making use of the above substitutions with respect to  $X$  and  $r$  and simple trigonometrical identities, after simple transformations we find the following:

$$(5.76) \quad c_0 \sqrt{\frac{2}{\kappa - 1}} \sin \beta \sin \delta_2 \leq e^G \leq c_0 \sqrt{\frac{2}{\kappa - 1}} \sin \beta \sin \delta_1$$

where

$$(5.77) \quad \delta_1 = -\sqrt{\rho_1} \arctan\left(\sqrt{\frac{\rho_1 \sin^2 \beta - 1}{\rho_1(1 - \sin^2 \beta)}}\right) + \arctan\left(\sqrt{\frac{\rho_1 \sin^2 \beta - 1}{1 - \sin^2 \beta}}\right) + f_0$$

$$\delta_2 = \sqrt{\rho_1} \arctan\left(\sqrt{\frac{\rho_1 \sin^2 \beta - 1}{\rho_1(1 - \sin^2 \beta)}}\right) - \arctan\left(\sqrt{\frac{\rho_1 \sin^2 \beta - 1}{1 - \sin^2 \beta}}\right) + f_0.$$



On the other hand we have  $0 \leq e^G \leq \chi_1$  and so

$$(5.78) \quad 0 \leq e^G \leq \sqrt{\frac{2}{\kappa-1}} c_0 \sin \beta.$$

Thus summing up we have

$$(5.79) \quad c_0 \sqrt{\frac{2}{\kappa-1}} \sin \beta \max[0, \sin \delta_2] \leq c_0 \sqrt{\frac{2}{\kappa-1}} \sin \beta \sin \delta_1$$

$$0 \leq e^{G(\beta_0)} \leq c_0 \sqrt{\frac{2}{\kappa-1}}$$

where  $\sin^2 \beta_0 = \frac{1}{\rho}$ ,  $\beta = \sqrt{\frac{\kappa-1}{2}}(R + c_1)$  and  $\beta$  obeys the inequality (5.51). This is a set of all restrictions for an arbitrary function  $G$ .

Finally we derive the following expressions which are an exact solution of the system of equations (5.44).

$$(5.80) \quad \begin{aligned} R &= \Psi(\lambda_1 x + \lambda_2 y + \lambda_3 z) \\ \varphi_1 &= \eta_1 e^G \sin K(R) \\ \varphi_2 &= -\eta_1 e^G \cos K(R) \\ \varphi_3 &= \eta_2 \left( \frac{2c_0^2}{\kappa-1} \sin^2 \left( \sqrt{\frac{\kappa-1}{2}}(R + c_1) \right) - e^{2G} \right)^{1/2} \\ K(R) &= \varepsilon \int_{R_0}^R \frac{\left\{ \frac{4c_0^4}{(\kappa-1)^2} \sin^4 \left( \sqrt{\frac{\kappa-1}{2}}(R + c_1) \right) + \right.}{e^G \left( \frac{2c_0^2}{\kappa-1} \sin^2 \left( \sqrt{\frac{\kappa-1}{2}}(R + c_1) \right) - e^{2G} \right)^{1/2}} \\ &\quad \left. - \left[ c_0^2 \cos^2 \left( \sqrt{\frac{\kappa-1}{2}}(R + c_1) \right) + e^{2G} \left( 1 + \left( \frac{dG}{dR} \right)^2 \right) \right] \frac{2c_0^2}{(\kappa-1)} \sin^2 \left( \sqrt{\frac{\kappa-1}{2}}(R + c_1) \right) + \right. \\ &\quad \left. - \varepsilon e^{2G} \frac{dG}{dR} \frac{2c_0^2}{(\kappa-1)} \sin \left( \sqrt{2(\kappa-1)}(R + c_1) \right) \right\}^{1/2}}{dR' + c_2} \end{aligned}$$

$\lambda_1, \lambda_2, \lambda_3$  are given by the right-hand side of the expressions (5.44) where (5.47) and (5.48) should be substituted instead of  $\sin \alpha$  and  $\cos \alpha$ . The function  $\Psi \in C_0^\infty$  and it takes values only from one arbitrary interval given by the inequalities (5.51). The arbitrary function  $G$  has the implied conditions (5.73).

Let us also discuss the case  $\frac{dG}{dR} = 0$ ,  $G = G_0 = \text{const}$ . Here we have the following condition

$$(5.81) \quad \left| \frac{e^{G_0} \sin \beta \sqrt{\frac{2}{\kappa-1}}}{\left(\frac{\kappa+1}{\kappa-1} \sin^2 \beta - 1\right)^{1/2} \left(\frac{2c_0^2}{(\kappa-1)} \sin^2 \beta e^{2G_0}\right)^{1/2}} \right| \leq 1$$

From which we get:

$$(5.82) \quad e^{2G_0} \left( \frac{\kappa+3}{\kappa-1} \sin^2 \beta - 1 \right) \leq \frac{2c_0^2(\kappa+1)}{(\kappa-1)^2} \sin^4 \beta$$

where  $\frac{1}{\rho_1} \leq \sin \beta \leq 1$ .

$\frac{1}{\rho_1} = \frac{\kappa-1}{\kappa+1} > \frac{\kappa-1}{\kappa+3}$  so we have

$$(5.83) \quad e^{2G_0} \leq \frac{2c_0^2(\kappa+1)}{(\kappa-1)^2} \frac{\sin^4 \beta}{\left(\frac{\kappa+3}{\kappa-1} \sin^2 \beta - 1\right)}.$$

Thus

$$(5.84) \quad \begin{aligned} e^{2G_0} &\leq \frac{2c_0^2(\kappa+1)}{(\kappa-1)^2} \min_{\left(\frac{1}{\rho} \leq \sin^2 \beta \leq 1\right)} \left[ \frac{\sin^4 \beta}{\frac{\kappa+3}{\kappa-1} \sin^2 \beta - 1} \right] = \\ &= \frac{2c_0^2(\kappa+1)}{(\kappa-1)^2} \min \left[ g \left( \frac{2(\kappa-1)}{\kappa+3} \right), g(1) \right] = \frac{2c_0^2(\kappa+1)}{(\kappa-1)^2} \min \left[ \frac{4(\kappa-1)^2}{(\kappa+3)^2}, \frac{\kappa-1}{2} \right] \end{aligned}$$

where

$$g(\xi) = \frac{\xi^2}{\frac{\kappa+3}{\kappa-1} \xi - 1}.$$

Finally we have

$$(5.85) \quad e^{2G_0} \leq \frac{8c_0^2(\kappa+1)}{(\kappa+3)^2}.$$

On the other hand we have

$$(5.86) \quad 0 \leq e^G \leq \chi_1$$

so we have

$$(5.87) \quad 0 \leq e^{G_0} \leq \min \chi_1 = c_0 \sqrt{\frac{2}{\kappa + 1}}$$

$$(5.88) \quad 0 \leq e^{G_0} \leq 2c_0^2 \min \left[ \frac{1}{\kappa - 1}, \frac{4(\kappa + 1)}{(\kappa + 3)^2} \right].$$

For  $1 < \kappa \leq 1 + \sqrt{\frac{16}{3}}$  we have

$$(5.89) \quad \frac{1}{\kappa - 1} \geq \frac{4(\kappa + 1)}{(\kappa + 3)^2}$$

Thus finally

$$(5.90) \quad 0 \leq e^{G_0} \leq \frac{8c_0^2(\kappa + 1)}{(\kappa + 3)^2}$$

Therefore it is the only restriction for constant  $G_0$ . Inserting  $\frac{dG}{dR} = 0$  and  $G = G_0 = \text{const}$  obeying (5.90) into expressions (5.80), (5.14) and (5.48) we derive the solution for constant  $G$ .

Now let us consider the cases for  $K = 1$ . They are similar to those discussed above, but certain differences occur, which are pointed out here. Substituting into equation (2.3) the simple integral element  $\lambda^{(1)}$ ,  $K = 1$ , i.e.  $F_3$  (Table 1 Appendix A) we get the following system

of equations

$$(5.91a) \quad \frac{d\varphi_1}{dR} = \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_2} \left( -(\varphi_2 \sin \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \cos \alpha) + \right. \\ \left. + (-\varphi_2 \cos \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha) \operatorname{sh} \rho \right) + \frac{c\varphi_1}{\chi_1} \operatorname{ch} \rho$$

$$(5.91b) \quad \frac{d\varphi_2}{dR} = \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_2} \left( (\varphi_1 \sin \alpha - \frac{\varphi_2 \varphi_3}{\chi_1} \cos \alpha) + \right. \\ \left. + (\varphi_1 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha) \operatorname{sh} \rho \right) + \frac{c\varphi_2}{\chi_1} \operatorname{ch} \rho$$

$$(5.91c) \quad \frac{d\varphi_3}{dR} = \frac{\chi_2(\chi_1^2 - c^2)^{1/2}}{\chi_1} (\cos \alpha - \sin \alpha \operatorname{sh} \rho) + \frac{c\varphi_3}{\chi_1} \operatorname{ch} \rho$$

where  $\chi_1^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2$ ,  $\chi_2^2 = \varphi_1^2 + \varphi_2^2$  and  $\alpha$  and  $\rho$  are functions of  $\varphi_1, \varphi_2, \varphi_3$  i.e.  $R$ . We write down equations for  $\chi_1, \chi_2$  and  $\mu_0 = \frac{\varphi_1}{\varphi_2}$ .

$$(5.92) \quad \frac{d\chi_1}{dR} = c \operatorname{ch} \rho \geq c$$

$$\frac{d\chi_2}{dR} = \varepsilon \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_1} (\chi_1^2 - \chi_2^2)^{1/2} (\sin \alpha \operatorname{sh} \rho - \cos \alpha) + \frac{c\chi_2}{\chi_1} \operatorname{ch} \rho$$

$$\frac{d}{dR} \arctan \mu_0 = -\frac{(\chi_1^2 - c^2)^{1/2}}{\chi_2} (\sin \alpha + \cos \alpha \operatorname{sh} \rho)$$

where  $\varepsilon_1^2 = 1$  and  $c^2 = c_0^2 - \frac{\kappa-1}{2} \chi_1^2 \geq 0$

We shift the arbitrariness from  $\alpha$  and  $\rho$  to  $\chi_1$  and  $\chi_2$  assuming that  $\chi_1^2 = e^{2H}$ ,  $\chi_2^2 = e^{2G}$  and then we search for the restrictions for those functions. So we have

$$(5.93) \quad \operatorname{ch} \rho = \frac{1}{c} e^H \frac{dH}{dR} \quad , \quad \operatorname{sh} \rho = \varepsilon_2 \left( \frac{1}{c^2} e^{2H} \left( \frac{dH}{dR} \right)^2 - 1 \right)^{1/2}.$$

Regarding equations (5.92) as an algebraic equation for a function  $\sin \alpha$  and using equations (5.93) we achieve the following

$$(5.94) \quad A = ZB - \varepsilon_3 \sqrt{1 - Z^2}$$

where  $Z = \sin \alpha$ ,  $\varepsilon_3^2 = 1$  and

$$(5.95) \quad A = \frac{\varepsilon_1 e^{(H+G)} \left( \frac{dG}{dR} - \frac{dH}{dR} \right)}{(e^{2H} - c^2)^{1/2} (e^{2H} - e^{2G})^{1/2}}$$

$$B = \varepsilon_2 \left( \frac{1}{c^2} e^{2H} \left( \frac{dH}{dR} \right)^2 - 1 \right)^{1/2}$$

requiring that  $|Z| \leq 1$  and that  $\Delta$  of the quadratic equation

$$(5.96) \quad Z^2(B^2 + 1) - 2ABZ + (A^2 - 1) = 0$$

is non-negative, using the equations (5.96) and (5.7) we achieve the same restrictions for  $H$  and  $G$  i.e. (5.16), (5.31), (5.32) and (5.33).  $\sin \alpha$ ,  $\cos \alpha$ , are given by formula (5.38) whereas  $\text{ch } \rho$  and  $\text{sh } \rho$  are given by (5.39) or (5.93), in the case when  $G \neq \text{const}$ . In the case when  $G = G_0 = \text{const}$  we have the restriction (5.43) instead of (5.16). Also the restriction for the function  $\Psi$  is identical. The only thing that changes is the form of solution and it is the following:

$$(5.97) \quad \begin{aligned} \varphi_1 &= \eta_2 e^G \sin K(R) & \eta_1^2 &= \eta_2^2 = 1 \\ \varphi_2 &= \eta_2 e^G \cos K(R) \\ \varphi_3 &= \eta_1 \sqrt{e^{2H} - e^{2G}} \end{aligned}$$

where

$$K(R) = - \int_{R_0}^R \frac{(e^{2H} - c^2)^{1/2}}{e^G} (\sin \alpha + \cos \alpha \text{sh } \rho) dR' + c$$

$c = \text{const.}$

and  $R = \Psi(\lambda_1 x + \lambda_2 y + \lambda_3 z)$ .

$\lambda_1, \lambda_2, \lambda_3$  are given by the expressions for  $K = 1$  from Appendix A (Table 1), whereas  $\sin \alpha, \cos \alpha, \operatorname{ch} \rho$  and  $\operatorname{sh} \rho$  are given by the expressions written above. The only difference between (5.97) and (5.37) is the form of the function  $K(R)$ . All the arguments concerning the interpretation of the solution are still valid.

Now let us turn to the analysis of the simple element  $\overset{(2)}{\lambda}$ ,  $K = 1$  (i.e.  $F_4$ , see Table 1, Appendix A). Then from equation (2.3) we get the following system of equations

$$\begin{aligned}
 \frac{d\varphi_1}{dR} &= \frac{\varepsilon(\chi_1^2 - c^2)^{1/2}}{\chi_2} \left( -\varphi_2 \cos \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha \right) + \frac{c\varphi_1}{\chi_1} \\
 \frac{d\varphi_2}{dR} &= \frac{\varepsilon(\chi_1^2 - c^2)^{1/2}}{\chi_2} \left( \varphi_1 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha \right) + \frac{c\varphi_2}{\chi_1} \\
 \frac{d\varphi_3}{dR} &= -\varepsilon \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_1} \chi_2 \sin \alpha + \frac{c\varphi_3}{\chi_1}, \quad \varepsilon^2 = 1.
 \end{aligned}
 \tag{5.98}$$

By introducing (as before) variables  $\chi_1, \chi_2, \mu_0$  we get the following equations

$$\begin{aligned}
 \frac{d\chi_1}{dR} &= c \\
 \frac{d\chi_2}{dR} &= \varepsilon \frac{(\chi_1^2 - c^2)^{1/2}}{\chi_1} (\chi_1^2 - \chi_2^2)^{1/2} \sin \alpha + \frac{c\chi_2}{\chi_1} \\
 \frac{d}{dR} \arctan \mu_0 &= -\frac{\varepsilon(\chi_1^2 - c^2)^{1/2}}{\chi_2} \cos \alpha
 \end{aligned}
 \tag{5.99}$$

which are identical to (5.45). We shall not derive from that any other solutions different from (5.80).

According to Appendix A we can write down equations (2.3) for the element  $\overset{(1')}{\lambda}$  for  $K = 0$

i.e.  $F_{1'}$  (see Table 2, Appendix A)

$$\begin{aligned}
 \frac{d\varphi_1}{dR} &= \frac{\sqrt{\chi_1^2 - c^2}}{c\chi_2} \left( \varphi_2 \cos \beta - \frac{\varphi_1 \varphi_3}{\chi_1} \sin \beta \right) + \frac{\varphi_1}{\chi_1} \\
 \frac{d\varphi_2}{dR} &= -\frac{\sqrt{\chi_1^2 - c^2}}{c\chi_2} \left( \varphi_1 \cos \beta + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \beta \right) + \frac{\varphi_2}{\chi_1} \\
 \frac{d\varphi_3}{dR} &= \frac{\sqrt{\chi_1^2 - c^2}}{c\chi_1} \sin \beta + \frac{\varphi_3}{\chi_1}.
 \end{aligned}
 \tag{5.100}$$

Introducing variables  $\chi_1, \chi_2, \mu_0$  we get:

$$\begin{aligned}
 \frac{d\chi_1}{dR} &= 1 \\
 \frac{d\chi_2}{dR} &= -\varepsilon_1 \frac{\sqrt{\chi_1^2 - c^2}}{\chi_1} (\chi_1^2 - \chi_2^2)^{1/2} \sin \beta + \frac{\chi_2}{\chi_1} \\
 \frac{d}{dR} \arctan \mu_0 &= \frac{\sqrt{\chi_1^2 - c^2}}{c\chi_2} \cos \beta
 \end{aligned}
 \tag{5.101}$$

where of course

$$\varphi_3 = \varepsilon_1 (\chi_1^2 - \chi_2^2)^{1/2} \quad , \quad \varepsilon_1^2 = 1.$$

The first equation (5.101) is immediately integrated and we get

$$\chi_1 = R + c_1 \tag{5.102}$$

where  $c_1 = \text{const.}$  Because of the fact that

$$c_0^2 \frac{2}{\kappa + 1} \leq \chi_1^2 \leq c_0^2 \left( \frac{2}{\kappa - 1} \right) \tag{5.103}$$

we have a restriction for the variable  $R$

$$c_0^2 \left( \frac{2}{\kappa - 1} \right) \leq (R + c_1)^2 \leq c_0^2 \left( \frac{2}{\kappa - 1} \right) \tag{5.104}$$

Now we deal with the second equation of (5.101) and according to the procedure proposed here we shift the arbitrariness from  $\beta$  to  $\chi_2$ , assuming that  $\chi_2 = e^G$ . Then we find the restriction for the arbitrary function  $G$ . Thus, treating the second of equations (5.101) as an equation for a function  $\sin \beta$  we get:

$$(5.105) \quad \sin \beta = \varepsilon_1 \frac{1 - (R + c_1) \frac{dG}{dR}}{\left((R + c_1)^2 - e^{2G}\right)^{1/2} \left(\frac{\kappa+1}{2}(R + c_1)^2 - c_0^2\right)^{1/2}}$$

where of course

$$(5.106) \quad e^{2G} \leq (R + c_1)^2.$$

Obviously  $|\sin \beta| \leq 1$ , and this leads to the following differential inequality:

$$(5.107) \quad \left| \frac{1 - \frac{dG}{dR} (R + c_1)}{\left((R + c_1)^2 - e^{2G}\right)^{1/2} \left(\frac{\kappa+1}{2}(R + c_1)^2 - c_0^2\right)^{1/2}} \right| \leq 1.$$

Taking  $X = \frac{R + c_1}{e^G}$  and  $R + c_1 = Z$  we reduce (5.107) to the following form

$$(5.108) \quad \left| \frac{X^2 \frac{dX}{dZ}}{\left(\frac{\kappa+1}{2}Z^2 - c_0^2\right)^{1/2} (X^2 - 1)^{1/2}} \right| \leq 1.$$

Then by taking other independent and dependent variables

$$(5.109) \quad \eta = \frac{c_0^2}{8} \sqrt{\frac{2}{\kappa+1}} \left[ (\xi + \sqrt{\xi^2 - 1})^2 - \frac{1}{(\xi + \sqrt{\xi^2 - 1})^2} - 4 \ln(\xi + \sqrt{\xi^2 - 1}) \right]$$

$$(5.110) \quad t = \frac{1}{8} \left[ (X + \sqrt{X^2 - 1})^2 - \frac{1}{(X + \sqrt{X^2 - 1})^2} \right] + \frac{1}{2} \ln(X + \sqrt{X^2 - 1})$$

where

$$\xi = \frac{1}{c_0} \sqrt{\frac{\kappa+1}{2}} Z = \frac{1}{c_0} \sqrt{\frac{\kappa+1}{2}} (R + c_1)$$



we reduce (5.108) to the following convenient form

$$(5.111) \quad \left| \frac{dt}{d\eta} \right| \leq 1.$$

Obviously we have

$$(5.112) \quad \frac{d\eta}{dZ} = \left( \left( \frac{\kappa+1}{2} \right) Z^2 - c_0^2 \right)^{1/2}$$

$$(5.113) \quad \frac{dt}{dX} = \frac{X^2}{(X^2 - 1)^{1/2}}.$$

Let  $t(\eta_0) = t_0$ . Then from (5.111) we get

$$(5.114) \quad -(\eta - \eta_0) + t_0 \leq t \leq (\eta - \eta_0) + t_0 \quad , \quad \eta \geq \eta_0$$

From which we obtain

$$(5.115) \quad -(\eta - \eta_0) + t_0 \leq F(X) \leq (\eta - \eta_0) + t_0$$

where

$$(5.116) \quad F(X) = \frac{1}{8} \left[ (X + \sqrt{X^2 - 1})^2 - \frac{1}{(X + \sqrt{X^2 - 1})^2} \right] + \frac{1}{2} \ln(X + \sqrt{X^2 - 1})$$

$$(5.117) \quad \begin{aligned} & X \geq 1 \\ & \frac{dF}{dX} = \frac{X^2}{(X^2 - 1)^{1/2}} \geq 0 \end{aligned}$$

So the function  $F$  is an increasing one and consequently it possesses the inverse function  $Q$  such as:

$$(5.118) \quad Q(F(X)) = X \quad \text{for } X \geq 1$$

$$(5.119) \quad \begin{aligned} \overline{\mathcal{D}}F &= [0, +\infty) \quad , \quad \overline{\mathcal{D}}Q = [1, +\infty) \\ \mathcal{D}F &= [1, +\infty) \quad , \quad \mathcal{D}Q = [0, +\infty). \end{aligned}$$

We also have

$$(5.120) \quad F(1) = 0 \quad , \quad \lim_{X \rightarrow 1^+} \frac{dF}{dX} = +\infty$$

$$\lim_{X \rightarrow +\infty} F(X) = +\infty \quad , \quad \lim_{X \rightarrow +\infty} \frac{dF}{dX} = +\infty$$

and

$$(5.121) \quad \frac{d^2 F}{dX^2} = \frac{X(2X^2 - \frac{1}{2}X - 2)}{(X^2 - 1)^{3/2}}.$$

In the interval  $[1, +\infty)$   $\frac{d^2 F}{dX^2} = 0$ , for  $X_0 = \frac{1}{8}(1 + \sqrt{65}) > 1$  and there the function  $F$  has its point of inflection. Thus, from (5.115) we get

$$(5.122) \quad Q(-(\eta - \eta_0) + t_0) \leq \frac{R + c_1}{e^G} \leq Q((\eta - \eta_0) + t_0) \quad , \quad \eta \geq \eta_0$$

and  $t_0 \geq (\eta - \eta_0)$  (The function  $Q$  is defined only for a non-negative argument  $t$ ).

$$(5.123) \quad \eta_0 \leq \eta \leq t_0 + \eta_0 \quad , \quad t_0 \geq 0$$

Now let us examine the following function

$$(5.124) \quad \eta(\xi) = \frac{c_0^2}{8} \sqrt{\frac{2}{\kappa + 1}} \left[ (\xi - \sqrt{\xi^2 - 1})^2 - \frac{1}{(\xi + \sqrt{\xi^2 - 1})^2} - 4 \ln(\xi + \sqrt{\xi^2 - 1}) \right]$$

$$\text{for } 1 \leq \xi \leq \sqrt{\frac{\kappa + 1}{\kappa - 1}} = \sqrt{\rho_1}.$$

$\frac{d\eta}{d\xi} > 0$  for  $\xi > 1$  and  $\frac{d\eta}{d\xi} = 0$  for  $\xi = 1$ . Thus it is an increasing function

$$(5.125) \quad \lim_{\xi=1} \eta(\xi) = 0 \quad , \quad \lim_{\xi \rightarrow +\infty} \eta(\xi) = +\infty.$$

We conclude that in the interval  $[1, \sqrt{\rho_1}]$  we have:

$$(5.126) \quad 0 \leq \eta(\xi) \leq \frac{c_0^2}{8} \sqrt{\frac{2}{\kappa + 1}} \left[ (\sqrt{\rho_1} + \sqrt{\rho_1 - 1})^2 - \frac{1}{(\sqrt{\rho_1} + \sqrt{\rho_1 - 1})^2} - 4 \ln(\sqrt{\rho_1} + \sqrt{\rho_1 - 1}) \right].$$

It is apparent, that it is convenient to assume  $\eta_0 = 0$  and

$$(5.127) \quad t_0 = \frac{c_0^2}{8} \sqrt{\frac{2}{\kappa+1}} \left[ (\sqrt{\rho_1} + \sqrt{\rho_1 - 1})^2 - \frac{1}{(\sqrt{\rho_1} + \sqrt{\rho_1 - 1})^2} - 4 \ln(\sqrt{\rho_1} + \sqrt{\rho_1 - 1}) \right].$$

In such a situation we have a maximal interval for  $\eta$  and consequently for  $R$  as well.

So finally we have

$$(5.128) \quad t_0 = F(X(0))$$

which leads to

$$(5.129) \quad e^{2G} \Big|_{(R+c_1)=c_0} \sqrt{\frac{2}{\kappa-1}} = c_0^2 \left( \frac{2}{\kappa+1} \right) \frac{1}{Q^2(t_0)}.$$

On the other hand

$$(5.130) \quad e^{2G} \leq (R + c_1)^2$$

so finally we get:

$$(5.131) \quad \frac{(R + c_1)^2}{Q^2(t_0 - \eta)} \leq e^{2G} \leq \min \left( 1, \frac{1}{Q^2(t_0 + \eta)} \right) (R + c_1)^2.$$

Now we turn back to the system of equations (5.100) and write down the solution

$$(5.132) \quad \begin{aligned} \varphi_1 &= e^G \sin K(R) \\ \varphi_2 &= e^G \cos K(R) \\ \varphi_3 &= \varepsilon_1 \sqrt{(R + c_1)^2 - e^{2G}} \\ R &= \Psi(\lambda_1 x + \lambda_2 y + \lambda_3 z) \end{aligned}$$

where

$$K(R) = \varepsilon_2 \int_{R_0}^R \frac{\sqrt{((R' + c_1)^2 - e^{2G}) \left( \frac{\kappa+1}{2} (R' + c_1)^2 - c_0^2 \right) - \left( 1 - \frac{dG}{dR} (R' + c_1)^2 \right)}}{e^G \left( c_0^2 - \frac{\kappa-1}{2} (R' + c_1)^2 \right)^{1/2} ((R' + c_1)^2 - e^{2G})^{1/2}} dR' + c_2$$

$$(R_0 + c_1)^2 = c_0^2 \left( \frac{2}{\kappa+1} \right), \quad \varepsilon_2^2 = 1$$

and

$$\begin{aligned}
 (5.133) \quad \cos \beta &= \varepsilon_2 (1 - \sin^2 \beta)^{1/2} = \\
 &= \varepsilon_2 \sqrt{\frac{((R + c_1)^2 - e^{2G})\left(\frac{\kappa+1}{2}(R + c_1)^2 - c_0^2\right) - \left(1 - \frac{dG}{dR}(R + c_1)^2\right)}{((R + c_1)^2 - e^{2G})\left(\frac{\kappa+1}{2}(R + c_1)^2 - c_0^2\right)}}.
 \end{aligned}$$

Restrictions (5.131) are imposed on the function  $G$ , whereas  $\Psi$  is a smooth function taking its value from the following intervals

$$\begin{aligned}
 (5.134) \quad &\left[ c_0 \sqrt{\frac{2}{\kappa + 1}} - c_1, c_0 \sqrt{\frac{2}{\kappa - 1}} - c_1 \right] \\
 &\text{or} \\
 &\left[ -c_0 \sqrt{\frac{2}{\kappa - 1}} - c_1, -c_0 \sqrt{\frac{2}{\kappa + 1}} - c_1 \right]
 \end{aligned}$$

$\lambda_i$  are components of the covector  $\overset{(2')}{\lambda}$  for  $K = 0$  and according to Appendix A (5.133), (5.132) and (5.105) should be substituted to them.

In the case  $\lambda'$ ,  $K = 1$ , i.e.  $F_{2'}$ , and according to Table 2 of Appendix A we get equations (2.3) in the following form:

$$\begin{aligned}
 (5.135) \quad \frac{d\varphi_1}{dR} &= \frac{\sqrt{\chi_1^2 - c^2}}{c\chi_2} \left( -\varphi_2 \cos \omega - \frac{\varphi_1 \varphi_3}{\chi_1} \sin \omega \right) + \frac{\varphi_1}{\chi_1} \\
 \frac{d\varphi_2}{dR} &= \frac{\sqrt{\chi_1^2 - c^2}}{c\chi_2} \left( \varphi_1 \cos \omega - \frac{\varphi_2 \varphi_3}{\chi_1} \sin \omega \right) + \frac{\varphi_2}{\chi_1} \\
 \frac{d\varphi_3}{dR} &= \frac{\sqrt{\chi_1^2 - c^2}}{c\chi_2} + \frac{\varphi_3}{\chi_1}
 \end{aligned}$$

For  $\chi_1, \chi_2$  and  $\mu_0$  we get

$$\begin{aligned}
 (5.136) \quad & \frac{d\chi_1}{dR} = 1 \\
 & \frac{d\chi_2}{dR} = -\varepsilon_1 \frac{(\chi_1^2 - c^2)^{1/2} (\chi_1^2)^{1/2} (\chi_1^2 - \chi_2^2)^{1/2}}{c\chi_1} \sin \omega + \frac{\chi_2}{\chi_1} \\
 & \frac{d}{dR} \arctan \mu_0 = -\cos \omega \frac{\sqrt{\chi_1^2 - c^2}}{c\chi_2}.
 \end{aligned}$$

These equations are identical to (5.101) except for the sign in the third equation. Thus they lead to almost identical solutions with an identical restriction as before.

Finally we consider the last simple integral elements, i.e.  $F_{1''}, F_{2''}, F_{3''}$  (compare Table 3, Appendix A). However these elements do not lead us to a new solutions.

Let us also note that our equation (3.6) is invariant under permutation of  $\varphi_1, \varphi_2, \varphi_3$ . Thus we may obtain exact solutions by permutating  $\varphi_1, \varphi_2, \varphi_3$  in expressions (5.37), (5.80), (5.97) etc. The solutions break the permutation symmetry  $P_3$ . This is spontaneous symmetry breaking and it results from the acceptance of the form of the matrix  $B(\alpha, l)$  which does not posses this symmetry.

**THEOREM.** *There are simple waves for all simple elements collected in Appendix A. All details are given above*

Proof is also given above.

All described solutions have a gradient catastrophe on a certain hypersurface  $S$ . On this surface some shock waves can appear.

## 6. Gauge and Bäcklund transformation

In this section we define a gauge transformation and an analogue for simple waves.

Now we construct some geometrical structures for simple (Riemann) waves of (2.5). These structures establish relations between the Riemann waves of (2.5) and allow us to introduce nonlinear transformations connecting two Riemann waves (exact solutions of (2.5)). The nonlinear transformations are of gauge type, and may be treated as Bäcklund transformation [9, 10] for (2.5).

Specifically, let the matrix  $A = (a_{ij})$  have  $K$  eigenvalues  $\omega_i$ ,  $i = 1, 2, \dots, K$ , each of order  $l_i$ ,  $\sum_{i=1}^K l_i = n$ .

In this case we have a natural group acting on simple elements  $\lambda$ , i.e.  $\prod_{i=1}^K \otimes O(l_i)$ . Each of the groups  $O(l_i)$  acts on the coordinates  $V_j$ ,  $j = \sum_{r=1}^{i-1} l_r$ ,  $\sum_{r=1}^{i-1} l_r + 1$ ,  $\dots$ ,  $\sum_{r=1}^i l_r$  without destroying the diagonalization of matrix  $A$ .

It is easy to see that  $\alpha_1, \alpha_2, \dots, \alpha_m$  are parameters of the group  $\prod_{i=1}^K \otimes O(l_i)$ ,  $m = \sum_{i=1}^K \frac{1}{2} l_i(l_i - 1)$ .

Simultaneously we can associate the group  $O(p, q)$  with the cone

$$(6.1) \quad \sum_{j=1}^n \zeta_j V_j^2 = 0, \quad (\zeta_j \text{ is equal to one of } \omega_i).$$

Let

$$(6.2) \quad \zeta_j = \varepsilon_j |\zeta_j| \quad \text{where} \quad \varepsilon_j = \text{sgn } \zeta_j.$$

By transforming  $V_j$  to  $V'_j$ ,  $j = 1, 2, \dots, n$

$$(6.3) \quad V'_j = \sqrt{|\zeta_j|} V_j$$

we transform (6.1) into

$$(6.4) \quad \sum_{j=1}^n \varepsilon_j V_j'^2 = 0$$

i.e. into a canonical cone.

The group  $O(p, q)$  preserves a quadratic form

$$(6.5) \quad Q(V', V') = \sum_{i=1}^n \varepsilon_i V_i'^2$$

where  $p =$  number of integers  $\varepsilon_i$  equal to 1,  $q =$  number of integers  $\varepsilon_i$  equal to  $(-1)$  in the sum (6.5).

Obviously  $p + q = n$  (we assume that there exist no zero eigenvalues).

Notice that classes of simple elements, and, in consequence, simple waves, which are constructed according to section 2, are related to the choice of a concrete chain of subgroups  $O(p, q)$ . This chain ends on the two-element group  $\{e, -e\}$  or the trivial group  $\{e\}$  hence

$$(6.6) \quad O(p, q) \supset O(p_1, q_1) \supset O(p_2, q_2) \supset \cdots \supset \{e, -e\} \supset \{e\}$$

where for  $p_i, q_1, p_{i+1}, q_{i+1}$  we have the following relations either

$$(6.7) \quad \begin{array}{ccc} p_i = p_{i+1} & & p_i = p_{i+1} + 1 \\ & \text{or} & \\ q_i = q_{i+1} + 1 & & q_i = q_{i+1} \end{array}$$

$p_0 = p, q_0 = q$ .

In this way the dimension of the space in which the group operates diminishes to 1 according section 2. The choice of the sequence of subscripts  $j_0, j_1, \dots, j_{K-2}$  corresponds to one of the possible chains of subgroups (6.6). Thus with each simple element we can associate in a natural way the following group

$$(6.8) \quad L_i = \left[ \otimes \prod_{r=1}^K O(l_i) \right] \otimes O(p_i, q_i).$$

The origin of each factor of the simple product is, of course, different. In general, we associate with equation (2.5) the group

$$(6.9) \quad L = \left[ \otimes \prod_{r=1}^K O(l_i) \right] \otimes O(p, q).$$

The group  $L_i$  acts on a submanifold  $F_i \subset \mathcal{E}^*$  (the manifold of simple elements of a chosen class according to the classification from section 2). Since a simple element is a function of a point of the hodograph space  $\mathcal{H}$  (the space of values of the solutions of the equation) we may construct some natural fibre bundles associated with the equation. For every class of simple elements we have a fibre bundle  $P_i$  over the base space  $\mathcal{H}$  with structural group  $L_i$ , typical fibre  $F_i$  and projection  $\pi_i : P_i \rightarrow \mathcal{H}$ .

It is easy to see that  $\dim(L_i) = \dim(F_i)$  and for every simple element,  $\lambda \in F_i^*$  we have

$$(6.10) \quad \lambda = g \cdot \lambda_0, \quad \text{where } g \in L_i \text{ and } \lambda_0 \in F_i$$

$$\lambda_0 = \text{const.}$$

Taking a local section of  $P_i$  we get

$$(6.11) \quad \lambda(u) = g(\alpha(u)) \lambda_0, \quad u \in \mathcal{D} \subset \mathcal{H}$$

where  $\alpha$  is the set of all parameters of the group  $L_i$ . But in the case of simple waves we have the Riemann invariant  $R$  (parametrization in the hodograph space  $\mathcal{H}$ ). Thus we obtain a special structure, a bundle  $\Pi_i$  over the base space  $\mathcal{R}$  (a straight line of the Riemann invariant) with structural group  $L_i$ , typical fibre  $F_i$  and with a projection  $\bar{\Pi}_i : \Pi_i \rightarrow \mathcal{R}$ .

For every local section of  $\Pi_i$  we get

$$(6.12) \quad \lambda(R) = g(\alpha(R)) \lambda_0, \quad R \in V \subset \mathcal{R}.$$

Every local section  $f$  gives us a simple wave belonging to a chosen class of simple waves (simple elements). If we have two local sections  $f$  and  $g$  we have two different simple waves of the same type. If we change the section from  $f$  to  $g$ , we change functions  $\alpha(R)$  to  $\beta(R)$  and we get

$$(6.13) \quad g(\alpha(R)) = h(R)g(\beta(R)), \quad \lambda(\alpha(R)) = h(R)\lambda(\beta(R)), \quad h(R) \in L.$$

Thus we see that the action of the gauge group of  $L_i$  over the straight line  $\mathcal{R}$  (Riemann invariant) on a simple wave creates a new simple wave of the same type ("gauge group")



means that the parameters of  $L_i$  depend on  $R$ ). In section 5 we solved equation (5.1) using arbitrary functions  $\alpha(R)$ . We shift the freedom from the  $\alpha$ 's to new, more convenient functions and we should do this for the functions  $\alpha$  and  $\beta$  independently. For  $\alpha$  and  $\beta$  one gets algebraic (or transcendental) equations, which will express  $\alpha$  (or  $\beta$ ) in terms of new functions and their first derivatives with respect to  $R$ . The condition of solvability of the algebraic (or transcendental) equations provide us with restrictions for the new functions. Varying  $\alpha(R)$  to  $\beta(R)$  we change these new functions and their first derivatives. Thus we get the gauge transformation connecting two exact solutions (simple waves of the same type). This transformation is very similar to the classical Bäcklund transformation [9]. For the equation (3.1) we have the following situation

$$(6.14) \quad L = O(2) \otimes O(1, 2)$$

and we used the following chains of subgroups of  $O(1, 2)$

$$(6.15) \quad \begin{aligned} O(1, 2) &\supset \overset{(1)}{O}(1, 1) \supset \{e, -e\} \\ O(1, 2) &\supset O(2) \supset \{e, -e\} \\ O(1, 2) &\supset \overset{(2)}{O}(1, 1) \supset \{e, -e\}. \end{aligned}$$

All these chains correspond to the simple elements and simple waves that we examined and we obtain the following gauge groups

$$(6.16) \quad \begin{aligned} L_1 &= O(2) \otimes \overset{(1)}{O}(1, 1) \quad , \quad L_{1'} = O(2) \otimes O(2) \quad , \quad L_{2'} = O(2) \otimes \{e, -e\} \quad . \\ L_2 &= O(2) \otimes \{e, -e\} \quad , \quad L_{1''} = O(2) \otimes \overset{(2)}{O}(1, 1) \quad , \quad L_{2''} = O(2) \otimes \{e, -e\} \quad . \end{aligned}$$

The case with  $L_1 = O(2) \otimes O(2)$  is very interesting because we have simple elements corresponding to that group with only one arbitrary parameter. Two parameters of  $O(2) \otimes O(2)$  become one parameter of the diagonal group  $O(2)$ .

Now let us write down the explicit form of the action of the gauge group of functions  $H$  and  $G$  for the case (4.1). Let us suppose that there exists one exact solution — a simple wave with parameters  $\alpha$  and  $\rho$  and corresponding arbitrary functions  $H$  and  $G$ . In this case we have  $L_i = O(2) \otimes O(1, 1) = L_1$ ,  $\alpha$  is a parameter of  $O(2)$  and  $\rho$  of  $O(1, 1)$ .

We change functions  $\rho$  and  $\alpha$  into  $\rho + \Delta\rho$  and  $\alpha + \Delta\alpha$ . It is a change of gauge by means of functions  $\Delta\rho$  and  $\Delta\alpha$ . And we look at how  $G$  and  $H$  change. In this way we obtain the explicit action of the gauge group on the manifold of functions  $H$  and  $G$  and their first derivatives with respect to  $R$ .

We have

$$(6.17) \quad \text{ch } \rho = \frac{e^H}{c(e^H)} \frac{dH}{dR} \quad , \quad \text{ch } (\rho + \Delta\rho) = \frac{e^{H_1}}{c(e^{H_1})} \frac{dH_1}{dR} \quad ,$$

but

$$(6.18) \quad \text{ch}(\rho + \Delta\rho) = \text{ch}\rho \text{ch}\Delta\rho + \text{sh}\rho \text{sh}\Delta\rho.$$

Inserting (5.17) and (4.39) into (5.18) we get:

$$(6.19) \quad \frac{e^{H_1}}{c(e^{H_1})} \frac{dH_1}{dR} = \frac{e^H}{c(e^H)} \frac{dH}{dR} \text{ch}(\Delta\rho) + \varepsilon_2 \left( \frac{e^{2H}}{c^2(e^{2H})} \left( \frac{dH}{dR} \right)^2 - 1 \right)^{1/2} \text{sh}(\Delta\rho).$$

(6.19) is the nonlinear representation of the gauge group originating from  $O(1, 1)$  on the manifold of functions  $H$  and  $G$  and their first derivatives with respect to  $R$  ( $H, H_1, G, G_1, \alpha, \rho, \Delta\alpha, \Delta\rho$  are functions of  $R$ ).

Similarly we get for  $O(2)$  from (6.20)

$$(6.20) \quad \sin(\alpha + \Delta\alpha) = \sin\alpha \cos\Delta\alpha + \sin\Delta\alpha \cos\alpha$$

(6.21)

$$\begin{aligned}
& \varepsilon_1 \left\{ \frac{e^{G_1} \frac{d}{dR} (H_1 - G_1) \left( \frac{e^{2H_1}}{c_1^2} \left( \frac{dH_1}{dR} \right)^2 - 1 \right)^{\frac{1}{2}} - \left[ \left( \frac{dH_1}{dR} \right)^2 (e^{2H_1} - e^{2G_1}) \frac{(e^{2H_1} - c^2)}{c^2} + \right.}{\frac{1}{c_1^2} e^{H_1} \left( \frac{dH_1}{dR} \right)^2 (e^{2H_1} - e^{2G_1})^{\frac{1}{2}} \cdot (e^{2H_1} - c_1^2)^{\frac{1}{2}}} \right.} \\
& \quad \left. \left. - e^{2G_1} \left( \frac{d}{dR} (H_1 - G_1) \right)^2 \right]^{\frac{1}{2}} \right\} = \\
& = \cos(\Delta\alpha) \varepsilon_2 \left\{ \frac{e^G \frac{d}{dR} (H - G) \left( \frac{e^{2H}}{c^2} \left( \frac{dH}{dR} \right)^2 - 1 \right)^{\frac{1}{2}} - \left[ \left( \frac{dH}{dR} \right)^2 (e^{2H} - e^{2G}) \frac{(e^{2H} - c^2)}{c^2} + \right.}{\frac{1}{c^2} e^H \left( \frac{dH}{dR} \right)^2 (e^{2H} - e^{2G})^{\frac{1}{2}} \cdot (e^{2H} - c^2)^{\frac{1}{2}}} \right.} \\
& \quad \left. \left. - e^{2G} \left( \frac{d}{dR} (H - G) \right)^2 \right]^{\frac{1}{2}} \right\} + \\
& \sin(\Delta\alpha) \varepsilon_3 \left\{ 1 - \frac{e^{2G} \left( \frac{d}{dR} (H - G) \right)^2 \left( \frac{e^{2H}}{c^2} \left( \frac{dH}{dR} \right)^2 - 1 \right) + \left( \frac{dH}{dR} \right)^2 (e^{2H} - e^{2G}) (e^{2H} - c^2) \frac{1}{c^2}}{\frac{1}{c^4} e^{2H} \left( \frac{dH}{dR} \right)^4 (e^{2H} - e^{2G}) \cdot (e^{2H} - c^2)} + \right. \\
& \quad + \frac{e^{2G} \left( \frac{d}{dR} (H - G) \right)^2 + 2e^G \frac{d}{dR} (H - G) \left( \frac{e^{2H}}{c^2} \left( \frac{dH}{dR} \right)^2 - 1 \right)^{\frac{1}{2}} \cdot \left[ \left( \frac{dH}{dR} \right)^2 (e^{2H} - e^{2G}) (e^{2H} - c^2) \frac{1}{c^2} + \right.}{\frac{1}{c^4} e^{2H} \left( \frac{dH}{dR} \right)^4 (e^{2H} - e^{2G}) \cdot (e^{2H} - c^2)} \\
& \quad \left. \left. + e^{2G} \left( \frac{d}{dR} (H - G) \right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}} .
\end{aligned}$$

Restrictions on the functions  $H, H_1, G, G_1$  and the range of the parameter  $R$  given in section 5 have been already imposed (see section 5).

Relations (6.19) and (6.21) are analogues of the Bäcklund transformation for equation (6.19). At the same time this action is a certain representation of the gauge group (a local one) on the manifold of arbitrary functions and their first derivatives parametrizing the solution. In this way the Bäcklund transformation for the equation (6.19) is a nonlinear representation of the gauge group (a local one) which has originated from the group  $L_1$ .

From the geometrical point of view the solution which corresponds to the group  $L_2$  is very interesting. The restrictions have forced the Riemann invariant  $R$  to belong to an interval. But the function  $e^{2H}$ , which is the length of the velocity vector, has the same values on the edges of the interval. So it is possible to identify these edges. In this way we obtain a fibre bundle over a circle.

## 7. Equation for potential nonstationary flow of a compressible perfect gas (the second example of an application of the method)

This section is devoted to the equation for potential nonstationary flow of compressible perfect gas. We discuss the flat nonstationary flow of a compressible gas described by the potential of velocity and density

$$(7.1) \quad \ln \rho = -\Phi_t \quad , \quad \vec{v} = \vec{\nabla} \Phi \quad .$$

This assumption allows us to describe discontinuities of velocity and density as a change of the gauge of this potential on both sides of the surface of discontinuity. Eliminating the density  $\rho$  from the mass conservation law, by means of the Euler equation

$$\left( \frac{\partial}{\partial t} + \vec{v} \vec{\nabla} \right) \vec{v} = - \frac{\vec{\nabla} p}{\rho} = - \frac{c^2}{\rho} \vec{\nabla} \rho \quad ,$$

we get

$$\frac{\partial \ln \rho}{\partial t} + c^2 \operatorname{div} \vec{v} - \vec{v} \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \vec{\nabla}) \vec{v} \right) = 0 \quad .$$

Introducing here the potential according to equation (7.1) we find (References [1–3])

$$(7.2) \quad \Phi_{,tt} + 2(\Phi_{,x}\Phi_{,xt} + \Phi_{,y}\Phi_{,yt}) + 2\Phi_{,x}\Phi_{,y}\Phi_{,xy} + (\Phi_{,x}^2 - c^2)\Phi_{,xx} + (\Phi_{,y}^2 - c^2)\Phi_{,yy} = 0 ,$$

where the lowest indices are for partial derivatives,  $\Phi: D \subset R^3 \rightarrow R$ ,  $D$  an open set. The velocity of sound  $c^2$  is given by a variant of the (compressible) Bernoulli equation:

$$\begin{aligned} \frac{\partial \Phi}{\partial t} + \frac{1}{2} (\vec{\nabla} \Phi)^2 + \int \frac{dP}{\rho} &= \text{const} , \quad \text{modulo equation (7.1),} \\ (1 - c^2) \frac{\partial \Phi}{\partial t} + \frac{1}{2} \vec{v}^2 &= \text{const.} \end{aligned}$$

We consider the special situation when the velocity of sound is constant  $c^2 \approx c_0^2$  (see References 11 and 12).

The above treatment of the potential, nonsteady, compressible flow is not very well known for the class of flows considered. Moreover it transforms to the correct equation (see References 11 and 12) without redefinition of the potential as in Thompson's book [13]. Thus all hydrodynamical quantities  $(\vec{v}, \rho)$  are treated equally in this particular potential approach.

It is easy to notice that equation (7.2) is a hyperbolic equation of the second order. We are interested in finding solutions that can be described by means of the Riemann invariant method. According to the requirements of the method we transform equation (7.2) by introducing some new dependent variables into the quasilinear system of equations of the first order,

$$\begin{aligned} \phi_{0,t} + 2(\phi_1\phi_{0,x} + \phi_2\phi_{0,y}) + 2\phi_1\phi_2\phi_{1,y} + (\phi_1^2 - c^2)\phi_{1,x} + (\phi_2^2 - c^2)\phi_{2,y} &= 0 , \\ (7.3) \quad \phi_{0,x} - \phi_{1,t} &= 0 \quad , \quad \phi_{1,y} - \phi_{2,x} = 0 , \\ \phi_{0,y} - \phi_{2,t} &= 0 , \end{aligned}$$

where we introduce the notation

$$\begin{aligned} \Phi_{,t} &= \phi_0 , \quad \Phi_{,x} = \phi_1 , \quad \Phi_{,y} = \phi_2 , \\ (7.4) \quad \vec{v} &= (\phi_1(t, x, y), \phi_2(t, x, y)) = \\ \rho &= \exp(-\phi_0(t, x, y)) . \end{aligned}$$

Thus equation (7.2) is reduced to an undetermined system of four equations for three functions  $\phi_0, \phi_1, \phi_2$ .

It is convenient to do the following transformations:

$$(7.5) \quad t \rightarrow t' = c_0 t, \quad \phi_0 \rightarrow \phi'_0 = \phi_0 / c_0.$$

Then we can write (7.3) in the following form:

$$(7.6) \quad \begin{aligned} & \phi'_{0,t} + 2(\phi'_1 \phi'_{0,x} + \phi'_2 \phi'_{0,y}) + 2\phi'_1 \phi'_2 \phi'_{1,y} + (\phi'^2_1 - 1)\phi'_{1,x} + (\phi'^2_2 - 1)\phi'_{2,y} = 0, \\ & \phi'_{0,x} - \phi'_{1,t} = 0, \quad \phi'_{1,y} - \phi'_{2,x} = 0, \\ & \phi'_{0,y} - \phi'_{2,t} = 0. \end{aligned}$$

The field of velocity of flow  $v$  and the density  $\rho$  are then in the following form:

$$\begin{aligned} \vec{v} &= c_0(\phi'_1(c_0 t, x, y), \phi'_2(c_0 t, x, y)), \\ \rho &= \exp(-c_0 \phi'_0(c_0 t, x, y)). \end{aligned}$$

## 8. Simple integral elements

### (the second example of an application of the method)

In this section we calculate simple integral elements for the equation for potential nonstationary flow of a compressible perfect gas.

Let us write equation (7.2) using simple integral elements. We get

$$(8.1a) \quad \gamma^0 \lambda_0 + 2(\phi_1 \gamma^0 \lambda_1 + \phi_2 \gamma^0 \lambda_2) + 2\phi_1 \phi_2 \gamma^1 \lambda_2 + (\phi_2^2 - c^2) \gamma^2 \lambda_2 + (\phi_1^2 - c^2) \gamma^1 \lambda_1 = 0,$$

$$(8.1b) \quad \gamma^\mu \lambda_\nu - \gamma^\nu \lambda_\mu = 0, \quad \mu, \nu = 0, 1, 2.$$

From equation (8.1b) we find that the vector  $\gamma$  is proportional to the vector  $\lambda$ . Thus, by inserting  $\gamma \sim \lambda$  into equation (8.1a) we get a quadratic form with respect to  $\lambda_0, \lambda_1, \lambda_2$ :

$$(8.2) \quad \begin{aligned} Q(\lambda_0, \lambda_1, \lambda_2) &= \lambda_0^2 + 2\lambda_0(\phi_1 \lambda_1 + \phi_2 \lambda_2) + 2\phi_1 \phi_2 \lambda_1 \lambda_2 + \\ &+ (\phi_2^2 - c^2) \lambda_1^2 + (\phi_1^2 - c^2) \lambda_2^2 = 0. \end{aligned}$$

Now, we follow Reference [7] and section 4. We transform the quadratic form (8.2) to a canonical form and search for eigenvalues of matrix  $A$  (the matrix of the quadratic form  $Q$ ),

$$(8.3) \quad \det(A_{ij} - \mu\delta_{ij}) = 0 ,$$

where

$$A = \begin{pmatrix} 1 & \phi_1 & \phi_2 \\ \phi_1 & (\phi_1^2 - c^2) & \phi_1\phi_2 \\ \phi_2 & \phi_1\phi_2 & (\phi_1^2 - c^2) \end{pmatrix} .$$

We get an algebraic equation of third order with respect to the quantity  $\mu$ , i.e.

$$(c^2 + \mu)[ -\mu^2 + \mu(\phi_1^2 + \phi_2^2 - c^2 + 1) + c^2 ] = 0 .$$

The eigenvalues  $\mu$  are real,

$$(8.4) \quad \begin{aligned} \mu_1 &= -c^2 , \\ \mu_{2,3} &= \frac{1}{2}(\phi_1^2 + \phi_2^2 - c^2 + 1 \pm \sqrt{\Delta}) , \end{aligned}$$

where  $\Delta = (\phi_1^2 + \phi_2^2 - c^2 + 1)^2 + 4c^2 > 0$ . Thus the quadratic form (8.2) transforms to a canonical form:

$$(8.5) \quad Q(y, y) = -c^2 y_1^2 + \frac{1}{2}(\phi_1^2 + \phi_2^2 - c^2 + 1 + \sqrt{\Delta}) y_2^2 + \frac{1}{2}(\phi_1^2 + \phi_2^2 - c^2 + 1 - \sqrt{\Delta}) y_3^2 .$$

According to References [7] and [8] and section 4 we parametrize the covector  $\lambda$ . To do this we search for a parametric equation of (8.5).

Let us suppose that  $y_1 \neq 0$ . Then equation (8.5) may be written in the form

$$(8.6) \quad \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 2c^2 , \quad \text{where} \quad X = \frac{y_2}{y_1} , \quad Y = \frac{y_3}{y_1} ,$$

and

$$(8.7) \quad \begin{aligned} a^2 &= (\phi_1^2 + \phi_2^2 - c^2 + 1 + \sqrt{\Delta})^{-1} , \\ b^2 &= (\sqrt{\Delta} - \phi_1^2 - \phi_2^2 + c^2 - 1)^{-1} . \end{aligned}$$

If  $y_1 = 0$ ,  $y_2 \neq 0$  and  $y_3 \neq 0$ , then equation (8.5) is

$$(8.8) \quad (v^2 - c^2 + 1 + \sqrt{\Delta}) y_2^2 + (v^2 - c^2 + 1 - \sqrt{\Delta}) y_3^2 = 0 ,$$

where  $v^2 = \varphi_1^2 + \varphi^2 = \bar{v}^2$ . Equation (8.6) is the equation of a hyperbola.

Thus we write in a parametric form

$$(8.9) \quad X = \sqrt{2} \, ac \, \text{ch } \tau, \quad Y = \sqrt{2} \, bc \, \text{sh } \tau,$$

where  $\tau$  is an arbitrary function of  $\phi_i$ ,  $i = 0, 1, 2$ . Thus the covector  $\lambda$  (and consequently  $\gamma$ ) become

$$(8.10) \quad \gamma_1 \sim \lambda^1 = B \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = B \begin{pmatrix} 1 \\ \sqrt{2} \, ac \, \text{ch } \tau \\ \sqrt{2} \, bc \, \text{sh } \tau \end{pmatrix},$$

where  $B$  is an orthogonal matrix, which diagonalizes the matrix  $A$ , i.e.,

$$(8.11) \quad B^T A B = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad B^T = B^{-1}.$$

Matrix  $B$  is built from the eigen vectors of matrix  $A$  and takes the form

$$(8.12) \quad B = h \begin{pmatrix} 0 & c^2 + 1 - v^2 + \sqrt{\Delta} & c^2 + 1 - v^2 - \sqrt{\Delta} \\ -\phi_2 & 2\phi_1 & 2\phi_1 \\ \phi_1 & 2\phi_2 & 2\phi_2 \end{pmatrix},$$

where

$$h = [v^2 \{(c^2 + 1 - v^2 + \sqrt{\Delta})^2 + 4v^2\} \times \{(c^2 + 1 - v^2 - \sqrt{\Delta})^2 + 4v^2\}]^{1/2}.$$

Inserting (8.8) and (8.12) into (8.10) we get a covector  $\lambda$ .

In the case with  $y_1 = 0$  we get

$$(8.13) \quad X = y_3/y_2 = \varepsilon b/a, \quad \varepsilon^2 = 1.$$

and finally we obtain

$$(8.14) \quad \gamma_2 \sim \lambda^2 = B \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} = B \begin{pmatrix} 0 \\ 1 \\ \varepsilon b/a \end{pmatrix}.$$



Inserting (8.8) and (8.12) into (8.14) we derive a simple element. According to References [7] and [8] we consider first the case with  $y_2 \neq 0$  and then with  $y_3 \neq 0$ . We may introduce the following coordinate systems:

$$X = \frac{y_1}{y_2} , \quad \frac{y_3}{y_2} , \quad \text{or} \quad X = \frac{y_1}{y_3} , \quad Y = \frac{y_2}{y_3} .$$

Proceeding as before we get the following simple elements:

$$(8.15) \quad \gamma_3 \sim \lambda^3 = B \begin{pmatrix} x \\ 1 \\ y \end{pmatrix} = B \begin{pmatrix} (1/\sqrt{2}ac) \cos \tau \\ 1 \\ (b/a) \sin \tau \end{pmatrix}$$

and

$$(8.16) \quad \gamma_4 \sim \lambda^4 = B \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = B \begin{pmatrix} (1/\sqrt{2}bc) \text{sh } \tau \\ (a/b) \text{sh } \tau \\ 1 \end{pmatrix} .$$

Let us consider the cases  $y_2 = 0$  or  $y_3 = 0$ . We get

$$X = y_3/y_1 = \varepsilon bc\sqrt{2} \quad \text{or} \quad X = y_2/y_1 = \varepsilon ac\sqrt{2} .$$

This time we obtain the following simple elements:

$$(8.17) \quad \gamma_5 \sim \lambda^5 = B \begin{pmatrix} 1 \\ 0 \\ x \end{pmatrix} = B \begin{pmatrix} 1 \\ 0 \\ \varepsilon bc\sqrt{2} \end{pmatrix}$$

and

$$(8.18) \quad \gamma_6 \sim \lambda^6 = B \begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} = B \begin{pmatrix} 1 \\ \varepsilon ac\sqrt{2} \\ 0 \end{pmatrix} .$$

Thus we get six kinds of simple elements that will be used for the construction of solutions, i.e. simple waves and their interactions, the so-called double and multiple waves. All of those simple elements are presented in Appendix B.

THEOREM. *All simple elements of the equation (7.2) are  $\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5$ .*

Proof has been given above.

## 9. Simple waves (the second example of an application of the method)

This section is devoted to simple waves for the equation of potential nonstationary flow for a compressible perfect gas.

We have the following cases (see [14]).

### A. Case I ( $\gamma_1 \sim \lambda^1$ — see Appendix B)

The simple wave, according to the considerations of section 2, is reduced here to fulfilling conditions (2.3) and (2.4). Inserting the simple integral element (8.10) into equation (2.3), we get

$$(9.1a) \quad \frac{d\phi_0}{dR} = c\sqrt{2} \left[ \frac{(c^2 + 1 - v^2 + \sqrt{\Delta})}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} \cos \tau + \frac{(c^2 + 1 - v^2 - \sqrt{\Delta})}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \sin \tau \right],$$

$$(9.1b) \quad \frac{d\phi_1}{dR} = -\phi_2 + 2c\sqrt{2}\phi_1 \left[ \frac{\text{ch } \tau}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} + \frac{\text{sh } \tau}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \right],$$

$$(9.1c) \quad \frac{d\phi_2}{dR} = \phi_1 + 2c\sqrt{2}\phi_2 \left[ \frac{\text{ch } \tau}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} + \frac{\text{sh } \tau}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \right],$$

where  $v^2 = \phi_1^2 + \phi_2^2$ ,  $\Delta = (v^2 - c^2 + 1)^2 + 4c^2$ . We are interested here in solving the system (7.3) with respect to the potential of the velocity field  $v = (\phi_1, \phi_2)$  and density

$\rho = \exp(-\phi_0)$ . Now let us assume that the expression in the square brackets in equation (9.1b) and (9.1c) is a smooth function of  $v^2$ , i.e.

$$(9.2) \quad \frac{\text{ch } \tau}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} + \frac{\text{sh } \tau}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} = f(v^2) .$$

That means  $\tau = \tau(v^2) = \tau(R)$ .

In this case we are able to find a solution in closed form. We introduce the quantity  $\mu_0 = \phi_1/\phi_2$  and divide equation (9.1b) [resp. equation (9.1c)] by  $\phi_1$  (resp.  $\phi_2$ ). Then we subtract both sides and finally we integrate and obtain

$$(9.3) \quad \mu_0 = \tan(c_1 - R) , \quad c_1 = \text{const.}$$

Then we introduce the quantity  $v^2 = \phi_1^2 + \phi_2^2$ . Thus, by multiplying equation (9.1b) by  $\phi_1$  and (9.1c) by  $\phi_2$  summing both sides, and then integrating we have

$$(9.4) \quad F(v^2) = \int_a^{v^2} \frac{dr}{cr f(r)} = 8\sqrt{2}R + c_2 ,$$

where  $a, c_2 = \text{const.}$

Now we assume there exists an inverse function  $G$  of  $F$  such that

$$G(F(r)) = r , \quad r > 0 ,$$

and we get

$$(9.5) \quad v^2 = G(s) , \quad \text{where } s = 8\sqrt{2}R + c_2 .$$

The function  $G$  is an arbitrary nonnegative function and it obeys the equation

$$(9.6) \quad \frac{dG}{ds} = \left( \frac{dF}{dv^2} \right)^{-1} = f(G(s)) .$$

From equations (9.3) and (9.5) we can calculate

$$(9.7) \quad \begin{aligned} \phi_1 &= \varepsilon G^{1/2}(8\sqrt{2}R + c_2) \sin(c_1 - R) , \\ \phi_2 &= \varepsilon G^{1/2}(8\sqrt{2}R + c_2) \cos(c_1 - R) , \quad \varepsilon^2 = 1 . \end{aligned}$$

Inserting (9.7) into (9.2) and simultaneously using relations (9.4) and (9.6) and then introducing  $G = e^{2H}$  we obtain

$$(9.8) \quad \left. \frac{dH}{ds} \right|_{s=8\sqrt{2}R+c_2} = \frac{1}{2c} \left[ \frac{\text{ch } \tau}{(1 + e^{2H} - c^2 + \sqrt{\Delta})^{1/2}} + \frac{\text{sh } \tau}{(c^2 - 1 - e^{2H} + \sqrt{\Delta})^{1/2}} \right],$$

$$\Delta = (e^{2H} - c^2 + 1)^2 + 4c^2.$$

where  $H$  is an arbitrary function of  $s$ . So, it is obvious that if the function  $\tau$  is given, then the function  $H$  is given as well and vice versa. But the function  $H$  is more convenient for parametrizing the simple element. Since we regard the quantity  $\tau$  as given, we are led to solving an ordinary differential equation with respect to  $H$ . Thus, from equation (9.8) we get

$$e^\tau = \alpha/2\sqrt{\Delta},$$

where

$$(9.9) \quad \alpha = 4 \frac{dH}{ds} + 2 \left( 8 \left( \frac{dH}{ds} \right)^2 + e^{2H} + 1 - c^2 \right)^{1/2}$$

and

$$(9.10) \quad \text{ch } \tau = \frac{\alpha^2 + 4\Delta}{4\alpha\sqrt{\Delta}}, \quad \text{sh } \tau = \frac{\alpha^2 - 4\Delta}{4\alpha\sqrt{\Delta}}.$$

As in [7] we search for a restriction of the function  $H$ . It is obvious that it must be

$$(9.11) \quad \begin{aligned} (1) \quad & 8 \left( \frac{dH}{ds} \right)^2 + e^{2H} + 1 - c^2 \geq 0, \\ (2) \quad & \alpha > 0. \end{aligned}$$

Condition (2) is easily satisfied by assuming that  $dH/ds \geq 0$ . If  $c = \text{const}$ , we may substitute  $c = 1$  and both conditions are always satisfied [cf. section 7, equations (7.5) and (7.6)]. Hence, the simple wave corresponding to the simple element (8.12) has the form

$$(9.12) \quad \begin{aligned} \vec{v} &= \varepsilon \exp[H(8\sqrt{2} + c_2)] (\sin(c_1 - R), \cos(c_1 - R)), \\ \varepsilon^2 &= 1. \end{aligned}$$

The function  $R = R(t, x, y)$  is to be understood in the context of expression (2.3) which means that the three-dimensional vector  $\vec{\nabla}R(t, x, y)$  is proportional to the vector  $\lambda^1$ . Then

$$\begin{aligned}
 R = & \Psi \left( \frac{\sqrt{2}c}{4\alpha\sqrt{\Delta}} \left[ \frac{2 - e^{2H} + \sqrt{\Delta}}{(e^{2H} + \sqrt{\Delta})^{\frac{1}{2}}} (\alpha^2 + 4\Delta) + \frac{2 - e^{2H} - \sqrt{\Delta}}{(-e^{2H} + \sqrt{\Delta})^{\frac{1}{2}}} (\alpha^2 - 4\Delta) \right] t + \right. \\
 (9.13) \quad & + \varepsilon \exp H \left\{ \left[ 4\sqrt{2} \frac{dH}{ds} \sin(c_1 - R) - \cos(c_1 - R) \right] x + \right. \\
 & \left. \left. + 4\sqrt{2} \left[ \frac{dH}{ds} \cos(c_1 - R) + \sin(c_1 - R) \right] y \right\} \right) .
 \end{aligned}$$

The density  $\rho$  is given by

$$(9.14) \quad \rho = \rho_0 \exp(-\phi_0) , \quad \rho_0 = \text{const},$$

where  $\phi_0$  is

$$\begin{aligned}
 \phi_0 = & \int_{R_0}^R \frac{c \, dR'}{\sqrt{2}(4dH/ds + [2(8(dH/ds)^2 + e^{2H} - 1 - c^2)]^{\frac{1}{2}})((e^{2H} - c^2 + 1)^2 + 4c^2)^{\frac{1}{2}}} \times \\
 & \times \left\{ \frac{c^2 + 1 - e^{2H} + ((e^{2H} - c^2 + 1)^2 + 4c^2)^{\frac{1}{2}}}{[e^{2H} - c^2 + 1 + ((e^{2H} - c^2 + 12 + 4c^2))^{\frac{1}{2}}]} \left( 16 \left( \frac{dH}{ds} \right)^2 + 4 \frac{dH}{ds} \left[ 2 \left( 8 \left( \frac{dH}{ds} \right)^2 + e^{2H} - 1 - c^2 \right) \right]^{\frac{1}{2}} + \right. \right. \\
 & + 2e^{4H} + 2c^4 + 1 - 4c^2 e^{2H} + 5e^{2H} + 8c^2 \Big) + \frac{c^2 - 1 - e^{2H} - ((e^{2H} - c^2 + 12 + 4c^2)^{\frac{1}{2}})}{[c^2 - 1 - e^{2H} + ((e^{2H} - c^2 + 1)^2 + 4c^2)^{\frac{1}{2}}]} \times \\
 & \left. \left. \times \left( 16 \left( \frac{dH}{ds} \right)^2 + 4 \frac{dH}{ds} \left[ 2 \left( 8 \left( \frac{dH}{ds} \right)^2 + e^{2H} - 1 - c^2 \right) \right]^{\frac{1}{2}} - 2e^{4H} - 2c^4 - 3 + 4c^2 e^{2H} - 3e^{2H} - 5c^2 \right) \right\} .
 \end{aligned}$$

(9.15)

The function  $H$  is a function of  $s = 8\sqrt{2}R + c_2$  and is arbitrary ( $c_2 = \text{const}$ ) and  $\frac{dH}{ds} \geq 0$ .

Now we introduce the quantity

$$(9.16) \quad \delta = \lambda_0 + \vec{v} \cdot \vec{\lambda} ,$$

which has a physical interpretation as the velocity of a moving wave with respect to the medium, whereas  $\lambda_0$  is the local velocity of the wave. In our case  $\delta$  and  $\lambda_0$  take the following form:

$$(9.17) \quad \lambda_0 = \frac{\sqrt{2}}{4\alpha\sqrt{\Delta}} \left[ \frac{c^2 - e^{2H} + \sqrt{\Delta}}{(e^{2H} + \sqrt{\Delta})^{1/2}} (\alpha^2 + 4\Delta) + \frac{2 - e^{2H} - \sqrt{\Delta}}{(-e^{2H} + \sqrt{\Delta})^{1/2}} (\alpha^2 - 4\Delta) \right],$$

$$\delta = \lambda_0 + \exp(2H) \left\{ \left[ 4\sqrt{2} \frac{dH}{ds} \sin(c_1 - R) - \cos(c_1 - R) \right] \sin(c_1 - R) + \right.$$

$$\left. + \left[ 4\sqrt{2} \frac{dH}{ds} \cos(c_1 - R) + \sin(c_1 - R) \right] \cos(c_1 - R) \right\}.$$

## B. Case II ( $\gamma_2 \sim \lambda^2$ — see Appendix B)

A simple wave corresponding to the simple integral element (8.14) may be found by integrating the following system of equations:

$$(9.17a) \quad \frac{d\phi'_0}{dR} = (2 - v^2 + \sqrt{\Delta})(\sqrt{\Delta} - v^2)^{1/2} + (2 - v^2 - \sqrt{\Delta})(v^2 + \sqrt{\Delta})^{1/2},$$

$$(9.17b) \quad \frac{d\phi'_1}{dR} = 2\phi'_1[2(\sqrt{\Delta} + 2)]^{1/2},$$

$$(9.17c) \quad \frac{d\phi'_2}{dR} = 2\phi'_2[2(\sqrt{\Delta} + 2)]^{1/2},$$

where  $v^2 = \phi_1'^2 + \phi_2'^2$ ,  $\Delta = v^4 + 4$ . We assume here that the velocity of sound  $c = 1$ . By dividing both sides of equations (9.1b) and (9.1c) and integrating we get

$$(9.18) \quad \phi'_1 = c_1 \phi'_2, \quad c_1 = \text{const.}$$

Now we calculate the quantity  $v = |\vec{v}|$ :

$$(9.19) \quad v = [(1/(c_2 - 2\sqrt{2}R)^2 - 2)^2 - 4]^{1/4}, \quad c_2 = \text{const.}$$

Hence we obtain

$$(9.20) \quad \begin{aligned} \phi'_1 &= \frac{\varepsilon c_1}{\sqrt{c_1^2 + 1}} \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{1/4}, \\ \phi'_2 &= \frac{\varepsilon}{\sqrt{c_1^2 + 1}} \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{1/4}, \end{aligned}$$

$$\varepsilon^2 = 1.$$

Substituting (9.19) into (9.1a) and then integrating, we have

$$(9.21) \quad \phi'_0 = -6 \left[ \sqrt{2}(c_2 - 2\sqrt{2}R) + \arctan \frac{1 - (1 - 2(c_2 - 2\sqrt{2}R)^2)^{1/2}}{\sqrt{2}(c_2 - 2\sqrt{2}R)} \right].$$

Thus a simple wave corresponding to the simple element (8.14) has the following form:

$$(9.22) \quad \vec{v} = \frac{\varepsilon}{\sqrt{c_1^2 + 1}} \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{1/4} (c_1, 1),$$

$$\rho = \rho_0 \exp \left[ 6\sqrt{2}R - \arctan \frac{1 - (1 - 2(c_2 - 2\sqrt{2}R)^2)^{1/2}}{\sqrt{2}(c_2 - 2\sqrt{2}R)} \right].$$

The dependent variable  $R$ , i.e., the Riemann invariant, is given explicitly by the formula

$$(9.23) \quad \begin{aligned} R = \Psi & \left( \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} \right) \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right) \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} + \\ & + \left( 4 - \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} - \frac{1}{(c_2 - 2\sqrt{2}R)^2} \right) \left( \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} + \right. \\ & \left. + \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^{\frac{1}{2}} \Big] t + \frac{2\varepsilon}{(c_1^2 + 1)^{\frac{1}{2}}} \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{4}} \times \\ & \times \left\{ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right) \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} \right)^{\frac{1}{2}} + \\ & + \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 + \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} \right)^{\frac{1}{2}} \Big\} (c_1 x + y), \end{aligned}$$

where  $\Psi$  is an arbitrary smooth (at least of class  $C^2$ ) function of one variable. The solution (9.22) with condition (9.23) describes one-dimensional, nonstationary flow that goes in the direction  $(c_1, 1)$ . It is worth mentioning that the solution is defined everywhere except of  $R_0 = c_2/2\sqrt{2}$ . The quantities  $\lambda_0$  and  $\delta$  (the total velocity of the wave and the velocity of the wave with respect to the medium) are as follows:

$$\begin{aligned} \lambda_0 = & \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} \right) \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 - \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} \right)^{\frac{1}{2}} + \right. \\ & \left. + \left( 4 - \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} - \frac{1}{(c_2 - 2\sqrt{2}R)^2} \right) \left( \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} + \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right) \right], \\ \delta = & \lambda_0 + 2\varepsilon(c_1^2 + 1)^{\frac{1}{2}} \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} \left\{ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 - \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} \right)^{\frac{1}{2}} + \right. \\ & \left. + \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 + \left[ \left( \frac{1}{(c_2 - 2\sqrt{2}R)^2} - 2 \right)^2 - 4 \right]^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\}. \end{aligned} \quad (9.24)$$

Moreover we have a restriction:

$$(9.25) \quad (1/(c_2 - 2\sqrt{2}R)^2 - 2)^2 - 4 \geq 0 .$$

This inequality may be easily solved and we obtain

$$R \geq (c_2 - 2)/2\sqrt{2} \quad \text{or} \quad R \leq (2 + c_2)/2\sqrt{2} .$$

### C. Case III ( $\gamma_3 \sim \lambda^3$ — see Appendix B)

A simple wave corresponding to simple integral elements (8.18) may be found by integrating the following system of ordinary differential equations:



$$\frac{d\phi_0}{dR} = (c^2 + 1 - v^2 + \sqrt{\Delta}) + (c^2 + 1 - v^2 - \sqrt{\Delta}) \times \frac{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \sin \tau,$$

$$\frac{d\phi_1}{dR} = \frac{-\phi_2}{c\sqrt{2}}(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2} \cos \tau + 2\phi_1 \left[ 1 + \left( \frac{v^2 - c^2 + 1 + \sqrt{\Delta}}{c^2 - 1 - v^2 + \sqrt{\Delta}} \right)^{1/2} \sin \tau \right],$$

$$\frac{d\phi_2}{dR} = \frac{\phi_1}{c\sqrt{2}}(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2} \cos \tau + 2\phi_2 \left[ 1 + \left( \frac{v^2 - c^2 + 1 + \sqrt{\Delta}}{c^2 - 1 - v^2 - \sqrt{\Delta}} \right)^{1/2} \sin \tau \right],$$

(9.26)

where  $v^2 = \phi_1^2 + \phi_2^2$ ,  $\Delta = (v^2 - c^2 + 1)^2 + 4c^2$ .

To solve the equations it is convenient to introduce a new variable  $\mu_0 = \phi_1/\phi_2$ . For quantities  $\mu_0$  and  $v^2$  we obtain the following system of equations:

$$(9.27) \quad \frac{dv^2}{dR} = \left[ 1 + \left( \frac{v^2 - c^2 + 1 + \sqrt{\Delta}}{c^2 - 1 - v^2 + \sqrt{\Delta}} \right)^{1/2} \sin \tau \right],$$

$$(9.28) \quad \frac{d}{dR} \arctan \mu_0 = \frac{(\sqrt{\Delta} - v^2 + c^2 - 1)^{1/2}}{c\sqrt{2}} \cos \tau,$$

where  $\tau$  is an arbitrary function of  $R$ . It is convenient to substitute

$$(9.29) \quad v^2 = e^{2H}$$

and  $H$  parametrizes the simple element (8.15).

We find the restrictions for  $H$ . Substituting (9.29) into (9.27) we have

$$(9.30) \quad \frac{1}{2} \left| \left( \frac{dH}{dR} - 2 \right) \left( \frac{c^2 - 1 - e^{2H} + \sqrt{\Delta}}{e^{2H} - c^2 + 1 + \sqrt{\Delta}} \right) \right| = |\sin \tau| \leq 1 .$$

Using the relations between trigonometrical functions we calculate  $\cos \tau$  and substitute into

(9.28 )

(9.31)

$$\frac{d}{dR} \arctan \mu_0 = \varepsilon_1 \frac{(\sqrt{\Delta} - e^{2H} + c^2 - 1)^{1/2}}{c\sqrt{2}} \times \left[ \pm \frac{1}{4} \left( \frac{dH}{dR} - 2 \right)^2 \frac{(c^2 - 1 - e^{2H} + \sqrt{\Delta})}{(e^{2H} - c^2 + 1 + \sqrt{\Delta})} \right]^{1/2},$$

$$\varepsilon_1^2 = 1.$$

The differential inequality (9.30 ) leads us to the solution for which the length of the vector  $\vec{v}$  is constant, i.e.  $H = H_0 = \text{const.}$  Then by inserting the quantity  $H = H_0$  into (9.27 ) and (9.28 ) and integrating we finally get

$$\vec{v} = ce^{H_0} (\varepsilon_2 \sin(K(R) + c_1), \cos(K(R) + c_1)) ,$$

$$\rho = \rho_0 \exp[-2c(R - R_0)(2 - e^{2H_0})] ,$$

$$K(R) = \varepsilon_1 \left[ \frac{e^{2H_0} [(e^{4H_0} + 4)^{1/2} - e^{2H_0}]}{(e^{4H_0} + 4)^{1/2} + e^{2H_0}} \right] (R - R_0) ,$$

(9.32)

$$\varepsilon_1^2 = \varepsilon_2^2 = 1 ,$$

$$R = \Psi(2c(2 - e^{2H_0})t + e^{H_0} \{ [4\varepsilon_2 \sin(K(R) + c_1) - \varepsilon_1 e^{H_0} \cos(K(R) + c_1)]x + [4 \cos(K(R) + c_1) + \varepsilon_1 \varepsilon_2 e^{H_0} \sin(K(R) + c_1)]y \}.$$

The quantities  $\lambda_0$  and  $\delta$  are

$$\lambda_0 = 2c(2 - e^{2H_0}) ,$$

$$\delta = 2c(2 - e^{2H_0}) + 4e^{2H_0} ,$$

(9.33)

and  $c$  is the velocity of sound.

#### D. Case IV ( $\gamma_5 \sim \lambda^5$ — see Appendix B)

A simple wave corresponding to the simple integral element (8.17) may be found by

integrating the following system of ordinary differential equations:

$$(9.34a) \quad \frac{d\phi'_0}{dR} = \frac{\varepsilon_1 \sqrt{2}(2 - v^2 + \sqrt{\Delta})}{(\sqrt{\Delta} - v^2)^{1/2}},$$

$$(9.34b) \quad \frac{d\phi'_1}{dR} = -\phi'_2 + 2\phi'_1 \frac{\varepsilon_1 \sqrt{2}}{(\sqrt{\Delta} - v^2)^{1/2}},$$

$$(9.34c) \quad \frac{d\phi'_2}{dR} = +\phi'_1 + 2\phi'_2 \frac{\varepsilon_1 \sqrt{2}}{(\sqrt{\Delta} - v^2)^{1/2}},$$

where  $\Delta = v^4 + 4$ ,  $v^2 = \phi_1'^2 + \phi_2'^2$ , and we have assumed that the velocity of sound equals 1. By dividing equation (9.34b) by  $\phi'_1$  and equation (9.34c) by  $\phi'_2$  and subtracting both sides of them and integrating we obtain

$$(9.35) \quad \phi'_1/\phi'_2 = \tan(c_1 - R), \quad c_1 = \text{const.}$$

Then we calculate  $v^2$ . We get

$$(9.36) \quad \begin{aligned} F_1(v^2) &= \frac{2}{3}((v^4 + 4)^{1/2} - v^2)^{3/2} + 4\sqrt{2} \arctan\left(\frac{(v^4 + 4)^{1/2} - v^2}{2}\right)^{1/2} + \\ &+ 2\sqrt{2} \ln \frac{((v^4 + 4)^{1/2} - v^2)^{1/2} + \sqrt{2}}{((v^4 + 4)^{1/2} - v^2)^{1/2} - \sqrt{2}} = 4\sqrt{2}R + c_2, \end{aligned}$$

$$c_2 = \text{const.}$$

The function  $F_1$  is monotone and consequently it has an inverse function  $G_1$ . Thus we have

$$(9.37) \quad v^2 = G_1(4\sqrt{2}R + c_2).$$

The domain of  $G_1$  is  $(-\infty, +\infty)$  and its range is  $(0, +\infty)$ .

From equations (9.35), (9.37), and (9.34a) we get

$$\phi'_1 = \varepsilon_2 G_1^{1/2} (4\sqrt{2}R + c_2) \sin(c_1 - R) ,$$

$$\phi'_2 = \varepsilon_2 G_1^{1/2} (4\sqrt{2}R + c_2) \cos(c_1 - R) ,$$

(9.38)

$$\phi'_0 = \varepsilon_2 \sqrt{2} \int_{R_0}^R \frac{(2 - G_1 + (G_1^2 + 4)^{1/2})}{((G_1^2 + 4)^{1/2} - G_1)^{1/2}} dR' + c_3 ,$$

$$c_3 = \text{const} ,$$

where  $G_1(F_1(x)) = x$  and  $G_1$  is a function of  $4\sqrt{2}R' + c_2$  (see Fig. 5).

Thus the simple wave corresponding to the simple element (8.17) is

$$\vec{v} = \varepsilon_2 c G_1^{1/2} (4\sqrt{2}R + c_2) (\sin(c_1 - R), \cos(c_1 - R)) ,$$

$$(9.39) \quad \rho = \rho_0 \exp\left(-\varepsilon_2 \sqrt{2} \int_{R_0}^R \frac{(2 - G_1 + (G_1^2 + 4)^{1/2})}{((G_1^2 + 4)^{1/2} - G_1)^{1/2}} dR'\right)$$

$$\rho_0 = \text{const} .$$

Moreover one can perform the integration in the second formula of (9.39) getting

(9.39')

$$\rho = \rho'_0 \exp\left(-4\varepsilon_2 \left(2 \ln G_1 - G_1 + (4 + G_1^2)^{1/2} \cdot \left(1 - \frac{2}{G_1}\right)\right)\right) \Big|_{s=4\sqrt{2}R+c_2}$$

$$\rho'_0 = \text{const} .$$

The dependent variable  $R$ , (i.e., Riemann invariant) is given in an implicit form

$$(9.40) \quad R = \Psi\left(\left[\frac{\varepsilon_1 \sqrt{2}(2 - G_1 + (G_1^2 + 4)^{1/2})}{((G_1^2 + 4)^{1/2} - G_1)^{1/2}}\right] ct + \right.$$

$$\left. + \varepsilon_2 G_1^{1/2} \left\{ \left[ 2 \sin(c_1 - R) \frac{\varepsilon_1 \sqrt{2}}{((G_1^2 + 4)^{1/2} - G_1)^{1/2}} - \cos(c_1 - R) \right] x + \right. \right.$$

$$\left. \left. + \left[ 2 \cos(c_1 - R) \frac{\varepsilon_1 \sqrt{2}}{((G_1^2 + 4)^{1/2} - G_1)^{1/2}} + \sin(c_1 - R) \right] y \right\} \right) ,$$

where  $\Psi$  is an arbitrary smooth (at least of  $C^2$  class) function of one variable. The quantities  $\lambda_0, \delta$ , i.e., respectively a local wave velocity and velocity of a moving wave with respect to the medium, equal

$$(9.41) \quad \begin{aligned} \lambda_0 &= \frac{c\varepsilon_1\sqrt{2}(2 - G_1 + (G_1^2 + 4)^{1/2})}{((G_1^2 + 4)^{1/2} - G_1)^{1/2}}, \\ \delta &= \frac{\varepsilon_1\sqrt{2}(2 + G_1 + (G_1^2 + 4)^{1/2})}{((G_1^2 + 4)^{1/2} - G_1)^{1/2}}. \end{aligned}$$

### E. Case V ( $\gamma_4 \sim \lambda^4$ — see Appendix B)

A simple wave corresponding to a simple element (8.16) can be found by integration of the following system of equations:

$$(9.42) \quad \begin{aligned} \frac{d\phi_0}{dR} &= (c^2 + 1 - v^2 - \sqrt{\Delta}) \left( \frac{\sqrt{\Delta} + v^2 + c^2 - 1}{\sqrt{\Delta} + v^2 - c^2 + 1} \right)^{1/2} \text{ch } \tau + (c^2 + 1 - v^2 - \sqrt{\Delta}), \\ \frac{d\phi_1}{dR} &= \frac{-\phi_2}{c\sqrt{2}} (\sqrt{\Delta} - v^2 + c^2 - 1)^{1/2} \text{sh } \tau + 2\phi_1 \left[ 1 + \left( \frac{\sqrt{\Delta} - v^2 + c^2 - 1}{\sqrt{\Delta} + v^2 - c^2 + 1} \right)^{1/2} \text{ch } \tau \right], \\ \frac{d\phi_2}{dR} &= \frac{\phi_1}{c\sqrt{2}} (\sqrt{\Delta} - v^2 + c^2 - 1)^{1/2} \text{sh } \tau + 2\phi_2 \left[ 1 + \left( \frac{\sqrt{\Delta} - v^2 + c^2 - 1}{\sqrt{\Delta} + v^2 - c^2 + 1} \right)^{1/2} \text{ch } \tau \right], \end{aligned}$$

where  $v^2 = \phi_1^2 + \phi_2^2$ ,  $\Delta = (v^2 - c^2 + 1)^{1/2} + 4c^2$ .

We introduce new dependent variables  $v^2$  and  $\mu_0 = \phi_1/\phi_2$  and we get

$$(9.43) \quad \frac{1}{2} \frac{dv^2}{dR} = 2v^2 \left[ 1 + \left( \frac{v^2 - c^2 + 1 + \sqrt{\Delta}}{c^2 - 1 - v^2 + \sqrt{\Delta}} \right)^{1/2} \text{ch } \tau \right],$$

$$(9.44) \quad \frac{d}{dR} \arctan \mu_0 = \frac{(\sqrt{\Delta} - v^2 + c^2 - 1)^{1/2}}{c\sqrt{2}} \text{sh } \tau,$$

where  $\tau$  is an arbitrary function of  $R$ . It is convenient to substitute

$$(9.45) \quad v^2 = e^{2H}.$$

Thus, the simple element (8.16) may be parametrized by the function  $H$  instead of  $\tau$  and we find a restriction for  $H$ . By inserting (9.45) into (9.43) we find

$$(9.46) \quad 1 \leq \text{ch } \tau = \frac{1}{2} \left( \frac{dH}{dR} - 1 \right) \left( \frac{c^2 - 1 - e^{2H} + \sqrt{\Delta}}{e^{2H} - c^2 + 1 + \sqrt{\Delta}} \right)^{1/2}.$$

Using relations between hyperbolic functions we calculate the quantity  $\text{sh } \tau$  and then substitute it into equation (9.44) to get

$$(9.47) \quad \frac{d}{dR} \arctan \mu_0 = \frac{\varepsilon_1 (\sqrt{\Delta} - e^{2H} + c^2 - 1)^{1/2}}{c\sqrt{2}} \left[ 1 - \frac{1}{4} \left( \frac{dH}{dR} - 2 \right)^2 \frac{(\sqrt{\Delta} - e^{2H} + c^2 + 1)}{(\sqrt{\Delta} + e^{2H} - c^2 + 1)} \right]^{1/2},$$

$$\varepsilon_1^2 = 1.$$

The differential inequality (9.46) leads to a solution for which the function  $H$  is constant,  $H = H_0 = \text{const}$ . It means that the length of the vector  $\vec{v}$  is constant. Thus by substituting  $H = H_0$  into equations (9.42) and (9.47) and then integrating, we get

$$(9.48) \quad \begin{aligned} \phi_1' &= \varepsilon_2 e^{H_0} \sin(K(R) + c_1), \quad \varepsilon_2^2 = 1, \\ \phi_2' &= e^{H_0} \cos(K(R) + c_1), \end{aligned}$$

where

$$(9.49) \quad K(R) = \varepsilon_1 \left[ \frac{e^{2H_0} (\sqrt{\Delta} - e^{2H_0} + 2)}{(\sqrt{\Delta} + e^{2H_0})} \right]^{1/2} (R - R_0),$$

$$\Delta = e^{4H_0} + 4,$$

$$c_1 = \text{const},$$

and

$$(9.50) \quad \phi_0' = c_2 + [-(2 - e^{2H_0} + \sqrt{\Delta})(\sqrt{\Delta} - e^{2H_0})/(\sqrt{\Delta} + e^{2H_0}) + (2 - e^{2H_0} - \sqrt{\Delta})](R - R_0),$$

where  $c_2 = \text{const}$ .

Finally we have

$$(9.51) \quad \vec{v} = c e^{H_0} (\varepsilon 2 \sin(K(R) + c_1), \cos(K(R) + c_1)) ,$$

$$\rho = \rho_0 \exp\{c[(2 - e^{H_0} + \sqrt{\Delta})(\sqrt{\Delta} - e^{2H_0})/(\sqrt{\Delta} + e^{2H_0}) - (2 - e^{2H_0} - \sqrt{\Delta})](R - R_0)\},$$

$$\rho_0 = \text{const.}$$

The Riemann invariant  $R$  is given in an implicit form as

$$(9.52) \quad R = \Psi \left( \left[ (2 - e^{2H_0} - \sqrt{\Delta}) - (2 - e^{2H_0} \sqrt{\Delta}) \frac{\sqrt{\Delta} - e^{2H_0}}{\sqrt{\Delta} + e^{2H_0}} \right] ct + \right. \\ \left. + e^{H_0} \left\{ \left[ 4\varepsilon_2 \frac{e^{2H_0}}{\sqrt{\Delta} + e^{2H_0}} \sin(K(R) + c_1) - \frac{\varepsilon_1 (\sqrt{\Delta} - e^{2H_0})^{1/2} e^{H_0}}{(\sqrt{\Delta} + e^{2H_0})^{1/2}} \cos(K(R) + c_1) \right] x + \right. \right. \\ \left. \left. + \left[ 4 \frac{e^{2H_0}}{\sqrt{\Delta} + e^{2H_0}} \cos(K(R) + c_1) + \frac{\varepsilon_1 \varepsilon_2 (\sqrt{\Delta} - e^{2H_0})^{1/2}}{(\sqrt{\Delta} + e^{2H_0})^{1/2}} \sin(K(R) + c_1) \right] y \right\} \right),$$

where  $\Psi$  is an arbitrary smooth function of one variable. The local velocity of the wave and the velocity with respect to the medium are constant and given by

$$(9.53) \quad \lambda_0 = c \left[ (2 - e^{2H_0} - \sqrt{\Delta}) - (2 - e^{2H_0} + \sqrt{\Delta}) \frac{(\sqrt{\Delta} - e^{2H_0})}{(\sqrt{\Delta} + e^{2H_0})} \right], \\ \delta = \lambda_0 + \frac{4e^{4H_0}}{\sqrt{\Delta} + e^{2H_0}},$$

where  $c = \text{const}$  and is the velocity of sound.

**F. Case VI** ( $\gamma_6 \sim \lambda^6$  — see **Appendix B**)

A simple wave corresponding to the simple element (8.18) may be found by integration of the following system of equations :

$$\begin{aligned}
 (9.54) \quad \frac{d\phi'_0}{dR} &= \frac{\varepsilon_1 \sqrt{2}(2 - v^2 + \sqrt{\Delta})}{(v^2 + \sqrt{\Delta})^{1/2}} , \quad \varepsilon_1^2 = 1 , \\
 \frac{d\phi'_1}{dR} &= -\phi'_2 + 2\phi'_1 \frac{\varepsilon_1 \sqrt{2}}{(v^2 + \sqrt{\Delta})^{1/2}} , \\
 \frac{d\phi'_2}{dR} &= \phi'_1 + 2\phi'_2 \frac{\varepsilon_1 \sqrt{2}}{(v^2 + \sqrt{\Delta})^{1/2}} ,
 \end{aligned}$$

where  $\Delta = v^4 + 4$ ,  $v^2 = \phi_1'^2 + \phi_2'^2$ .

We assume that the velocity of sound equals 1 and introduce new variables  $\mu_0 = \phi'_1/\phi'_2$  and  $v^2$ . As before we integrate equations for these variables and we get

$$(9.55) \quad \mu_0 = \phi'_1/\phi'_2 = \tan(c_1 - R) , \quad c_1 = \text{const} ,$$

$$\begin{aligned}
 (9.56) \quad F_2(v^2) &= -2 \arctan\left(\frac{((v^4 + 4)^{1/2} - v^2)^{1/2} - \sqrt{2}}{2}\right)^{1/2} + 2((v^4 + 4)^{1/2} - v^2)^{1/2} + \\
 &+ \ln\left[\frac{((v^4 + 4)^{1/2} - v^2)^{1/2} - \sqrt{2}}{((v^4 + 4)^{1/2} - v^2)^{1/2} + \sqrt{2}}\right] = 2\varepsilon_1 \sqrt{2}R + c_2 , \quad c_2 = \text{const}.
 \end{aligned}$$

The function  $F_2$  is monotone. So, in the interval  $(0, +\infty)$  it possesses an inverse function  $G_2$  such as

$$(9.57) \quad G_2(F_2(x)) = x .$$

We get

$$v^2 = G_2(2\varepsilon_1 \sqrt{2}R + c_2) .$$



From (9.55), (9.57), and (9.54) we have

$$\begin{aligned}
 \phi'_1 &= \varepsilon_2 G_2^{1/2} (2\varepsilon_1 \sqrt{2}R + c_1) \sin(c_1 - R) , \\
 (9.58) \quad \phi'_2 &= \varepsilon_2 G_2^{1/2} (2\varepsilon_1 \sqrt{2}R + c_1) \cos(c_1 - R) , \\
 \phi'_0 &= \varepsilon_1 \sqrt{2} \int_{R_0}^R \frac{(2 - G_2 + (G_2^2 + 4)^{1/2})}{((G_2^2 + 4)^{1/2} + G_2)^{1/2}} dR' + c_3 , \quad c_3 = \text{const.}
 \end{aligned}$$

where the function  $G_2$  is given by a transcendental equation

$$\begin{aligned}
 (9.59) \quad & (2((c_1^2 + 4)^{1/2} - G_2))^{1/2} + \ln \left[ \frac{((G_2^2 + 4)^{1/2} - G_2)^{1/2} - \sqrt{2}}{((G_2^2 + 4)^{1/2} - G_2)^{1/2} + \sqrt{2}} \right] + \\
 & - 2 \arctan \left( \frac{(G_2^2 + 4)^{1/2} - G_2}{2} \right)^{1/2} = 2\varepsilon_1 \sqrt{2}R + c_2 .
 \end{aligned}$$

Thus a simple wave corresponding to the simple element (8.18) is

$$\begin{aligned}
 \vec{v} &= \varepsilon_2 G_2^{1/2} (2\varepsilon_1 \sqrt{2}R + c_2) (\sin(c_1 - R), \cos(c_1 - R)) , \\
 (9.60) \quad \rho &= \rho_0 \exp \left( -\varepsilon_1 \sqrt{2} \int_{R_0}^R \frac{(2 - G_2 + (G_2^2 + 4)^{1/2})}{((G_2^2 + 4)^{1/2} + G_2)^{1/2}} dR' \right) \\
 \rho_0 &= \text{const} .
 \end{aligned}$$

Moreover one can performe an integration in the second formula of equation (9.60) getting

$$\begin{aligned}
 (9.60') \quad \rho &= \rho'_0 \exp \left( -\frac{1}{4} \left( 2 \ln G_2 - G_2 + (4 + G_2^2)^{1/2} \left( 1 - \frac{2}{G_2} \right) \right) \right) \Big|_{s=2\varepsilon_1 \sqrt{2}R+c_2} \\
 \rho'_0 &= \text{const} .
 \end{aligned}$$

The dependent variable  $R$ , the Riemann invariant, is given in an implicit form as

$$\begin{aligned}
 R = & \Psi \left( \left[ \frac{\varepsilon_1 \sqrt{2} (2 - G_2 + (G_2^2 + 4)^{1/2})}{(G_2 + (G_2^2 + 4)^{1/2})^{1/2}} \right] ct + \right. \\
 (9.61) \quad & + \varepsilon_2 G_2^{1/2} \left\{ \left[ \frac{2\varepsilon_1 \sqrt{2}}{(G_2 + (G_2^2 + 4)^{1/2})^{1/2}} \sin(c_1 - R) - \cos(c_1 - R) \right] x + \right. \\
 & \left. \left. + \left[ \frac{2\varepsilon_1 \sqrt{2}}{(G_2 + (G_2^2 + 4)^{1/2})^{1/2}} \cos(c_1 - R) + \sin(c_1 - R) \right] y \right\} \right),
 \end{aligned}$$

where  $\Psi$  is an arbitrary function of one variable. The quantities  $\lambda_0$ ,  $\delta$ , which are, respectively, the local wave velocity and the velocity of the wave with respect to the medium, are given by

$$\begin{aligned}
 \lambda_0 &= \frac{c\varepsilon_1 \sqrt{2} (2 - G_2 + (G_2^2 + 4)^{1/2})}{(c_1 + (G_2^2 + 4)^{1/2})^{1/2}}, \\
 (9.62) \quad \delta &= \frac{\varepsilon_1 \sqrt{2} (2 + G_2 + (G_2^2 + 4)^{1/2})}{G_2 + (G_2^2 + 4)^{1/2}}.
 \end{aligned}$$

Thus these simple waves are the basis for searching for a wider class of solution, the so called double waves and multiple waves. The superposition of this type may be very interesting from the physical point of view and they will be considered in future papers.

It is interesting to notice that our calculations can be extended to the three-dimensional case, but this will cause very tedious and laborous algebra. It seems that the assumption of the constancy of the velocity of sound can be abandoned. However, we cannot use some mathematical tricks in the above calculations and we probably cannot get compact results.

**THEOREM.** *There are simple waves for simple elements described in Appendix B. All details are given above.*

The proof is also given above.

All described solutions have a gradient catastrophe on a certain hypersurface  $S$ . On this hypersurface some shock waves can appear.

## **Conclusions**

In this paper we formulate an improved Riemann invariant method and apply it for two important equations of gas dynamics. Using the method, we find several wide classes of exact solutions with interesting properties which have physical interpretations and some applications in gas dynamics.

**Appendix A — simple elements (the first example)**

**Table 1.** Covectors  $\lambda$ ,  $(F_1, F_2)$

$$\begin{aligned} \lambda_{(K=0)}^{(1)} &= \begin{pmatrix} \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left[ -\left( \varphi_2 \sin \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \cos \alpha \right) + \left( \varphi_2 \cos \alpha - \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha \right) \text{sh} \rho \right] + \frac{c \varphi_1}{\chi_1} \text{ch} \rho \\ \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left[ \left( \varphi_1 \sin \alpha - \frac{\varphi_2 \varphi_3}{\chi_1} \cos \alpha \right) - \left( \varphi_1 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha \right) \text{sh} \rho \right] + \frac{c \varphi_2}{\chi_1} \text{ch} \rho \\ \frac{\chi_2 (\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_1} (\cos \alpha + \sin \alpha \text{sh} \rho) + \frac{c \varphi_3}{\chi_1} \text{ch} \rho \end{pmatrix} \\ \lambda_{(K=0)}^{(2)} &= \begin{pmatrix} \frac{\varepsilon (\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left( \varphi_2 \cos \alpha - \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha \right) + \frac{c \varphi_1}{\chi_1} \\ \frac{-\varepsilon (\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left( \varphi_2 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha \right) + \frac{c \varphi_2}{\chi_1} \\ \frac{\varepsilon (\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_1} \chi_2 \sin \alpha + \frac{c \varphi_3}{\chi_1} \end{pmatrix} \end{aligned}$$

Covectors  $\lambda$ ,  $K = 1$ ,  $(F_3, F_4)$

$$\begin{aligned} \lambda_{(K=1)}^{(1)} &= \begin{pmatrix} \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left[ -\left( \varphi_2 \sin \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \cos \alpha \right) + \left( -\varphi_2 \cos \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha \right) \text{sh} \rho \right] + \frac{c \varphi_1}{\chi_1} \text{ch} \rho \\ \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left[ \left( \varphi_1 \sin \alpha - \frac{\varphi_2 \varphi_3}{\chi_1} \cos \alpha \right) + \left( \varphi_1 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha \right) \text{sh} \rho \right] + \frac{c \varphi_2}{\chi_1} \text{ch} \rho \\ \frac{\chi_2 (\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_1} (\cos \alpha - \sin \alpha \text{sh} \rho) + \frac{c \varphi_3}{\chi_1} \text{ch} \rho \end{pmatrix} \\ \lambda_{(K=1)}^{(2)} &= \begin{pmatrix} \frac{\varepsilon (\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left( -\varphi_2 \cos \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \sin \alpha \right) + \frac{c \varphi_1}{\chi_1} \\ \frac{\varepsilon (\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left( \varphi_1 \cos \alpha + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \alpha \right) + \frac{c \varphi_2}{\chi_1} \\ \frac{-\varepsilon (\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_1} \chi_2 \sin \alpha + \frac{c \varphi_3}{\chi_1} \end{pmatrix} \end{aligned}$$

**Table 2.** Covectors  $\lambda', K = 0$  ( $F_{1'}, F_{2'}$ )

$$\lambda_{(K=0)}^{(1')} = \begin{pmatrix} \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{c\chi_2} (\varphi_2 \cos \beta - \frac{\varphi_1 \varphi_3}{\chi_1} \sin \beta) + \frac{\varphi_1}{\chi_1} \\ -\frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{c\chi_1} (\varphi_1 \cos \beta + \frac{\varphi_2 \varphi_3}{\chi_1} \sin \beta) + \frac{\varphi_2}{\chi_1} \\ \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{c\chi_1} \sin \beta + \frac{\varphi_3}{\chi_1} \end{pmatrix}$$

Covectors  $\lambda', K = 1$

$$\lambda_{(K=1)}^{(1')} = \begin{pmatrix} \frac{(\chi_1 - c^2)^{\frac{1}{2}}}{c\chi_2} (-\varphi_2 \cos \omega - \frac{\varphi_1}{\chi_1} \sin \omega) + \frac{\varphi_1}{\chi_1} \\ \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{c\chi_2} (\varphi_1 \cos \omega - \frac{\varphi_2 \varphi_3}{\chi_1} \sin \omega) + \frac{\varphi_2}{\chi_1} \\ \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{c\chi_1} \sin \omega + \frac{\varphi_3}{\chi_1} \end{pmatrix}$$

**Table 3.** Covectors  $\lambda'', (F_{1''}, F_{2''}, F_{3''})$

$$\lambda_{(K=0)}^{(1'')} = \begin{pmatrix} \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left[ -\frac{\varphi_1 \varphi_3}{\chi_1} (\text{sh} \rho \cos \alpha + \sin \alpha) - \varphi_2 (\text{sh} \rho \sin \alpha + \cos \alpha) \right] + \frac{c\varphi_1}{\chi_1} \text{ch} \rho \\ \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left[ -\frac{\varphi_2 \varphi_3}{\chi_1} (\text{sh} \rho \cos \alpha + \sin \alpha) + \varphi_1 (\text{sh} \rho \sin \alpha + \cos \alpha) \right] + \frac{c\varphi_2}{\chi_1} \text{ch} \rho \\ \frac{(\chi_1 - c^2)^{\frac{1}{2}}}{\chi_1} \chi_2 (\text{sh} \rho \cos \alpha + \sin \alpha) + \frac{c\varphi_3}{\chi_1} \text{ch} \rho \end{pmatrix}$$

$$\lambda_{(K=1)}^{(1'')} = \begin{pmatrix} \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left[ -\frac{\varphi_1 \varphi_3}{\chi_1} (\text{sh} \rho \cos \alpha - \sin \alpha) - \varphi_2 (\text{sh} \rho \sin \alpha - \cos \alpha) \right] + \frac{c\varphi_1}{\chi_1} \text{ch} \rho \\ \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_2} \left[ \frac{\varphi_2 \varphi_3}{\chi_1} (-\text{sh} \rho + \varphi_1 (\cos \alpha + \text{sh} \rho \sin \alpha)) \right] + \frac{c\varphi_2}{\chi_1} \text{ch} \rho \\ \frac{(\chi_1^2 - c^2)^{\frac{1}{2}}}{\chi_1} \chi_2 (\text{sh} \rho \cos \alpha - \sin \alpha) - \frac{c\varphi_3}{\chi_1} \text{ch} \rho \end{pmatrix}$$

$$\stackrel{(2'')}{\lambda} = \begin{pmatrix} \frac{-\varepsilon(\chi_1^2 - c^2)^{\frac{1}{2}}}{c\chi_2} \left( \varphi_2 \sin \alpha + \frac{\varphi_1 \varphi_3}{\chi_1} \cos \alpha \right) + \frac{\varphi_1}{\chi_1} \\ \frac{\varepsilon(\chi_1^2 - c^2)^{\frac{1}{2}}}{c\chi_2} \left( \varphi_1 \sin \alpha - \frac{\varphi_2 \varphi_3}{\chi_1} \cos \alpha \right) + \frac{\varphi_2}{\chi_1} \\ \frac{\varepsilon(\chi_1^2 - c^2)^{\frac{1}{2}}}{c\chi_1} \chi_2 \cos \alpha + \frac{\varphi_3}{\chi_1} \end{pmatrix}$$

$$K = 0, 1$$

where  $\alpha, \rho, \beta, \omega$  are arbitrary smooth (at least of class  $C^2$ ) functions of  $\varphi_i$ ,  $i = 1, 2, 3$  and  $\chi_1^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2$ ,  $\chi_2^2 = \varphi_1^2 + \varphi_2^2$ ,  $\varepsilon^2 = 1$ .

## Appendix B — simple elements (the second example)

The covectors  $\lambda$  are

$$\lambda^1 = \begin{pmatrix} c\sqrt{2} \left[ \frac{(c^2 + 1 - v^2 + \sqrt{\Delta})}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} \text{ch}\tau + \frac{(c^2 + 1 - v^2 - \sqrt{\Delta})}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \text{sh}\tau \right] \\ -\phi_2 + 2c\sqrt{2}\phi_1 \left[ \frac{\text{ch}\tau}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} + \frac{\text{sh}\tau}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \right] \\ \phi_1 + 2c\sqrt{2}\phi_2 \left[ \frac{\text{ch}\tau}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} + \frac{\text{sh}\tau}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \right] \end{pmatrix}$$

$$\lambda^2 = \begin{pmatrix} (c^2 + 1 - v^2 + \sqrt{\Delta})(\sqrt{\Delta} - v^2 + c^2 - 1)^{1/2} + (c^2 + 1 - v^2 - \sqrt{\Delta})(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2} \\ 2\phi_1 [\sqrt{\Delta} - v^2 + c^2 - 1]^{1/2} + (v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2} \\ 2\phi_2 [\sqrt{\Delta} - v^2 + c^2 - 1]^{1/2} + (v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2} \end{pmatrix}$$

$$\lambda^3 = \begin{pmatrix} (c^2 + 1 - v^2 + \sqrt{\Delta}) + (c^2 + 1 - v^2 - \sqrt{\Delta}) \frac{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \sin \tau \\ \frac{-\phi_2}{c\sqrt{2}} (v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2} \cos \tau + 2\phi_1 \left[ 1 + \left( \frac{v^2 - c^2 + 1 + \sqrt{\Delta}}{c^2 - 1 - v^2 + \sqrt{\Delta}} \right)^{1/2} \sin \tau \right] \\ \frac{\phi_2}{c\sqrt{2}} (v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2} \cos \tau + 2\phi_2 \left[ 1 + \left( \frac{v^2 - c^2 + 1 + \sqrt{\Delta}}{c^2 - 1 - v^2 + \sqrt{\Delta}} \right)^{1/2} \sin \tau \right] \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} \frac{\varepsilon c\sqrt{2}(c^2 + 1 - v^2 + \sqrt{\Delta})}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \\ -\phi_2 + 2\phi_1 \frac{\varepsilon c\sqrt{2}}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \\ \phi_1 + 2\phi_2 \frac{\varepsilon c\sqrt{2}}{(c^2 - 1 - v^2 + \sqrt{\Delta})^{1/2}} \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} (c^2 + 1 - v^2 + \sqrt{\Delta}) \left( \frac{\sqrt{\Delta} - v^2 + c^2 - 1}{\sqrt{\Delta} + v^2 - c^2 + 1} \right)^{1/2} \text{ch}\tau + (c^2 + 1 - v^2 - \sqrt{\Delta}) \\ \frac{-\phi_2}{c\sqrt{2}} (\sqrt{\Delta} - v^2 + c^2 - 1)^{1/2} \text{sh}\tau + 2\phi_2 \left[ 1 + \left( \frac{\sqrt{\Delta} - v^2 + c^2 - 1}{\sqrt{\Delta} + v^2 - c^2 + 1} \right)^{1/2} \text{ch}\tau \right] \\ \frac{\varphi_1}{c\sqrt{2}} (\sqrt{\Delta} - v^2 + c^2 - 1)^{1/2} \text{sh}\tau + 2\varphi_2 \left[ 1 + \left( \frac{\sqrt{\Delta} - v^2 + c^2 - 1}{\sqrt{\Delta} + v^2 - c^2 + 1} \right)^{1/2} \text{ch}\tau \right] \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} \frac{\varepsilon c \sqrt{2} (c^2 + 1 - v^2 + \sqrt{\Delta})}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} \\ -\phi_2 + 2\phi_1 \frac{\varepsilon c \sqrt{2}}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} \\ \phi_1 + 2\phi_2 \frac{\varepsilon c \sqrt{2}}{(v^2 - c^2 + 1 + \sqrt{\Delta})^{1/2}} \end{pmatrix}$$

where  $\tau$  is an arbitrary (at least of class  $C^2$ ) function of  $\phi_i$ ,  $i=0, 1, 2$  and  $v^2 = \phi_1^2 + \phi_2^2$ ,  
 $\Delta = (v^2 - c^2 + 1)^2 + 4c^2$ .



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